GRK 2240 Workshop: Complex reflection groups
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Abstract

A complex reflection of a vector space V is an element $g \in \mathrm{GL}(V)$ of finite order fixing a hyperplane point-wise. These groups include finite real reflection groups, also known as finite Coxeter groups. The theory of complex reflection groups has a wide range of applications including, for example, representation theory of reductive algebraic groups, moduli spaces and knot theory. The irreducible complex reflection groups have been classified by Shephard and Todd [5] in 1954. The classification includes the infinite series $G(m, p, n)$ and 34 exceptional groups denoted by $G_4, G_5, \ldots G_{37}$. Furthermore, the theory of complex reflection groups is deeply intertwined with invariant theory arising from work of Chevalley [1]. Our goal in this workshop is to introduce the groups and their classification as well as the connection to invariant theory. Our main sources will be [3] and [4]. I also recommend taking a look at the chapter by Geck and Malle in the Handbook of Algebra [2].

1 Preliminaries and the Shephard-Todd classification

The first talk should introduce (pseudo)-reflection groups and give an overview over the classifications of complex reflection groups. It follows sections 14-1, 14-2 and and 15-1 of [3]. State the definitions of a reflection and reflection group and explain the table on page 154 as well as Example 2. Explain what we understand under an irreducible reflection group and a pseudo-reflection.
Explain the basics of the classification of irreducible complex-reflection groups by Shephard and Todd. First define imprimitive and primitive groups and define and explain the series of groups $G(m, p, n)$. It might be helpful to read the beginning of chapter 2 in [4]. State Proposition 2.10 and Theorem 2.14 of [4] as one fact and prove the Proposition, if time allows. Furthermore, present some of the Examples [2.11, [4]]. Finally, explain how one broadly goes about to classify the rest of the groups and give one example of such an exceptional group.

2 Pseudo-reflection groups in other fields

The aim of the talk is to introduce a different look at pseudo-reflection groups and the connection to representation and character theory. Follow section 14-3 of [3] and explain the differences between the modular and non-modular case. Then, following section 15-2, explain how one can classify pseudo-reflections groups of general fields of characteristic 0 via the Theorem of Clark-Ewing, using and presenting the definitions and facts of representation theory as in [Appendix B, [3]]. On the topic of the field of definition, it might be helpful to take a look at sections 1.7 and 2.6 in [4]. Furthermore, detail what results we have in characteristic p as in [15-3, [3]].

3 Polynomial invariants and the Shephard-Todd-Chevalley Theorem

The aim of this section is to introduce the theory of polynomial invariants of finite reflection groups. It closely follows Chapter 3 of [4].
Start with Section 2.8 of [4] as motivation. Recall the definitions needed to define the algebra of invariants of a group. State Theorem 3.20 and finish with at least one example from [16-2, [3]]. Define the basic invariants for G and explain, why the set of basic invariants is not unique, but their number and degrees are uniquely determined by the group, referencing to the motivation in Section 2.8.

4 Characterization of reflection groups

The aim of this talk is to finish with the Theorem of Chevalley-Shephard-Todd-Bourbaki. Introduce the Poincaré series as in [17-1, [3]] and state some of the following examples. Then state Molien's Theorem and the first Example of [17-2, [3]]. Use the Theorem to prove the first part of Theorem 4.14 in [4]. Finally, state the main result in the form of Theorem 4.19. Comment on the modular case, as in the beginning of section 19 of [3].

References

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