Stable module categories

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GRK 2240 workshop: tensor triangular geometry summer semester 2021 June 15, 2021



Theorem (projective modules)

Recall that an R-module P is **projective** if one of the following equivalent conditions holds:

(i) We have the following uniersal lifting property:

$$M \xrightarrow{\xi'} \downarrow^{f} \downarrow^{g}$$

$$M \xrightarrow{p} N \longrightarrow 0$$

- (ii) The functor $\operatorname{Hom}_{\operatorname{Mod}(R)}(P,-)$ is exact.
- (iii) Any short exact sequence $0 \to M \to N \to P \to 0$ of R-modules splits.
- (iv) P is a direct summand of a free R-module.



Theorem (injective objects)

Recall that an R-module I is **injective** if one of the following equivalent conditions holds:

(i) We have the following universal lifting property:

$$0 \longrightarrow M \xrightarrow{i} N$$

$$f \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad N$$

$$I \longrightarrow M \longrightarrow N$$

- (iii) The functor $\operatorname{Hom}_{\operatorname{Mod}(R)}(-,I)$ is exact.
- (iii) Any short exact sequence $0 \to I \to M \to N \to 0$ of R-modules splits.



Definition (projective cover)

For an R-module M a *projective cover* is a pair (P,π) where P is projective and $\pi:P o M$ is an epimorphism.Define

$$\Omega(M, P, \pi) = \ker(\pi).$$

Definition (injective hull)

For an R-module M an injective hull is a pair (I,ι) where I is injective and $\iota:M o I$ is a monomorphism.Define

$$\Omega^{-1}(M, I, \iota) = \operatorname{coker}(\iota).$$



Theorem

(i) The category $\operatorname{Mod}(R)$ has enough projectives in the sense that for any projective covers then by Schanuel's lemma R-module M there is a projective cover.If (P,π) , (P',π') are two

$$\ker(\pi) \oplus P' \cong \ker(\pi') \oplus P.$$

(ii) The category $\mathrm{Mod}(R)$ has enough injectives in the sense that for any hulls then similarly R-module M there is an injetive hull.If (I,ι) , $(',\iota')$ are two injective

$$\operatorname{coker}(\iota) \oplus I' \cong \operatorname{coker}(\iota') \oplus I.$$



Definition (stable module category)

For R-modules M,N let

$$\operatorname{PHom}_{\operatorname{Mod}(R)}(M,N) = \left\{ f \in \operatorname{Hom}_{\operatorname{Mod}(R)}(M,N) \mid f : M \to P \to N \right\}$$

be the subspace of $\operatorname{Hom}_{\operatorname{Mod}(R)}(M,N)$ consisiting of morphisms which factor through a projective module

be the category with Then we define the *projectively stable module category* $\operatorname{PStMod}(R)$ of R to

$$ob(PStMod(R)) = ob(Mod(R)),$$

$$\operatorname{Hom}_{\operatorname{PStMod}(R)}(M, N) = \operatorname{Hom}_{\operatorname{Mod}(R)}(M, N) / \operatorname{PHom}_{\operatorname{Mod}(R)}(M, N).$$



Remark

Two R-modules M,N are isomorphic in $\mathrm{PStMod}(R)$ if and only if $M\oplus P\cong N\oplus Q$ for some projective modules P,Q. If $M\oplus P\cong N\cong Q$ via inverse isomorphisms f,g, then write

$$f = \begin{pmatrix} f_{MN} & f_{PN} \\ f_{MQ} & f_{PQ} \end{pmatrix}, \qquad g = \begin{pmatrix} g_{NM} & g_{QM} \\ g_{NP} & g_{QP} \end{pmatrix}.$$

Then the conditions

$$g \circ f = \mathrm{id}_{M \oplus P} = \begin{pmatrix} \mathrm{id}_M & 0 \\ 0 & \mathrm{id}_P \end{pmatrix}, \qquad f \circ g = \mathrm{id}_{N \oplus Q} = \begin{pmatrix} \mathrm{id}_N & 0 \\ 0 & \mathrm{id}_Q \end{pmatrix}$$

show $M\cong N$ via f_{MN} and g_{NM} . Still to shows: If $M\cong N$ there are projectives P,Q as above



Remark

- The procedure also works with injectives instead of projectives (one then module category IStMod(R). mods out $\operatorname{IHom}_{\operatorname{Mod}(R)}(M,N)$) and ends up with the *injectively stable*
- There is a tensor product on the stabe categories inherited from $\operatorname{Mod}(R)$. This will be used later in the workshop

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Definition (two functors)

In the projectively stable category we can define

$$\Omega(M) := \ker(\pi),$$

for some projective cover $\pi:P\to M$. In the injectively stable category we can define

$$\Omega^{-1}(M) := \operatorname{coker}(\iota)$$

for some injective hull $\iota: M \to I$.

cover π resp. the injective hull ι . We define moreover Note that these definitions are independent of the choice of the projective

$$\Omega^{n}(M) = \Omega(\Omega^{n-1}(M)), \qquad \Omega^{-n}(M) = \Omega^{-1}(\Omega^{-n+1}(M)).$$

Theorem (functorialty)

 $(\operatorname{IStMod}(R)).$ The assignment Ω (Ω^{-1}) defines and endofunctor on $\mathrm{PStMod}(R)$

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Let $f \in \operatorname{Hom}_{\operatorname{PStMod}(R)}(M,N)$ and consider projective covers of M and NWe exemplarily prove functoriality for Ω ; the proof for Ω^{-1} works similarly.

$$\begin{array}{cccc}
\Omega(M) & \longrightarrow P & \xrightarrow{\alpha} M \\
\Omega(f) \downarrow & & \uparrow & \downarrow \\
\Omega(N) & \longrightarrow Q & \xrightarrow{\beta} N
\end{array}$$

Since β is an epimorphism we can lift $f \circ \alpha$ to a map $\gamma: P \to Q$. Define

 $\Omega(f):=\gamma|_{\Omega(M)}$. This is well-defined: For two lifts γ_1,γ_2 the difference satisfies $\Omega(f_1) - \Omega(f_2)$ factors through P. $\operatorname{im}(\gamma_1 - \gamma_2) \subset \ker \beta = \Omega(N)$, i.e. $\gamma_1 - \gamma_2$ factors through $\Omega(N)$, hence

Definition

Let ${\cal G}$ be a finite group and k a field. Then

$$k[G] := \bigoplus_{g \in G} kg$$

is called the group algebra of G over k. If $H \subset G$ is a subgroup we have an M to H is the module M with multiplication $h \cdot m := \iota(h) \cdot m$ for $h \in k[H]$ inclusion $\iota: k[H] \hookrightarrow k[G]$. If further M is a k[G]-module, the restriction of



Example

- $oxed{1}$ If $R=\mathbb{Z}$ then $\mathrm{Mod}(R)=\mathrm{AbGrp}$ is the category of abelian groups \mathbb{Z} -modules are isomorphic in $\mathrm{PStMod}(\mathbb{Z})$ iff their torsion part coincides Then an R-module is projective iff it is free. Hence two finitely generated
- If R=k for a field then $\mathrm{Mod}(R)=\mathrm{Vect}_k$ is the category of k-vector spaces. Since any vector space is free hence projective in $\operatorname{PStMod}(k)$ all vector spaces are isomorphic to the trivial vector space
- 3 Let $G=\mathbb{Z}/2$, k a field of characteristic 2 and $R=k[G]=k[x]/x^2$. One can show that for any R-module M we have

M is projective \Leftrightarrow for all $0 \neq \alpha \in k$ the restriction $M|_{\langle 1+\alpha x\rangle}$ is free

where $\langle 1 + \alpha x \rangle \subset k[x]/x^2$ is cyclic of order 2 (Dade, 1978).



heorem

algebrak[G] is Frobenius Let R = k[G]. Then any R-module is projective iff it is injective, i.e.

Proof

non-degenrate and G-invariant pairing k[G] imes k[G] o k. This induces is free and thus projective $\Rightarrow k[G]$ is self-injective, i.e. injective as a k[G]-module, since $k[G]\cong k[G]^*$ $k[G]\cong k[G]^*$ as k-vector spaces and by G-invariance even as k[G]-modules The Kronecker symbol $\delta_{g,h}$ can be extended k[G]-linearly to a bilinear,

direct sums of injectives are injective (Bass-Papp) \Rightarrow free modules are injective

injective P projective $\Rightarrow P$ is direct summand of a free, i.e. injective module $\Rightarrow P$

Similarly I injective $\Rightarrow I^*$ is projective, i.e. direct summand of a free module $I\cong I^*\Rightarrow I$ is projective.



category of a Frobenius category. We write $\operatorname{StMod}(R) := \operatorname{PStMod}(R) = \operatorname{IStMod}(R)$ for the stable module

Corollary

 $\Omega(M) = \ker(P \to M)$ and $\Omega^{-1}(M) = \operatorname{coker}(M \to I)$). The funcotrs Ω and Ω^{-1} on $\operatorname{StMod}(k[G])$ are inverse to each other. (Recall:

Proof

for a projective cover $\alpha:P o M$ of some k[G]-module M we have an exact sequence

$$0 \to \Omega(M) \stackrel{\iota}{\hookrightarrow} P \stackrel{\alpha}{\to} M \to 0,$$

i.e. $M = \operatorname{coker}(\iota)$.

Similarly $\Omega(\Omega^{-1}(M)) = M$. Since P is also injective, $\Omega^{-1}(\Omega(M)) = \operatorname{coker}(\iota) = M$



Definition (triangles)

Let $\alpha \in \operatorname{Hom}_{\operatorname{StMod}(k[G])}(M,N)$. Then α induces a diagram

$$\begin{array}{ccc}
M & \xrightarrow{\alpha} & N \\
\downarrow^{I_M} & \xrightarrow{\overline{\alpha}} & \downarrow^{\beta} \\
\downarrow^{I_M} & \xrightarrow{\overline{\alpha}} & \uparrow^{\zeta} \\
\downarrow^{\overline{\iota}_M} & & & \uparrow^{\zeta} \\
\Omega^{-1}(M) & = & \Omega^{-1}(M)
\end{array}$$

triangle distinguished if it is isomorphic to a standard triangle. where the upper square is a pushout and γ exists by the pushout property of C_{α} . The triangle $M \to N \to C_{\alpha} \to \Omega^{-1}(M)$ is called *standard*. We call a



Theorem

 $\operatorname{StMod}(k[G])$ is a triangulated category with shift Ω^{-1} .

Proof

category (where injective and projectives coincide) is triangulated We will more generally show that the stable category of any Frobenius

- [T1] By construction distinguished triangles are closed under isomorphisms and any morphism indues a distinguished triangle
- stable category, i.e. $M o M o 0 o \Omega^{-1}(M)$ is a standard triangle. Moreover the identity $1_M:M\to M$ induces $C_{1_M}=I_M\cong 0$ in the
- $[\mathsf{T2}]$ Given a distinguished triangle $M o N o L o \Omega^{-1}(M)$ we need to show that $N \to L \to \Omega^{-1}(M) \to \Omega^{-1}(N)$ is distinguished. $M \xrightarrow{\alpha} N \to C_{\alpha} \to \Omega^{-1}(M)$. Restrict to standard triangles, i.e. we have a standard triangle



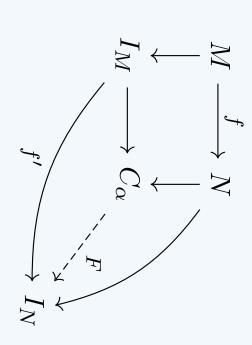
Strategy: "Add" some injective I to $\Omega^{-1}(M)$ (which is an isomorphism $f:M\to N$ consider as before and the shifted triangle is even standard (hence distinguished). For in $\operatorname{StMod}(k[G]))$ such that $\Omega^{-1}(M)\oplus I$ identifies with the pushout

$$0 \longrightarrow N \longrightarrow I_N \longrightarrow \Omega^{-1}(N) \longrightarrow 0$$

$$\uparrow \qquad \qquad \uparrow \\ \qquad \uparrow \\ \qquad \downarrow \\ f' \qquad \qquad \uparrow \\ \Omega^{-1}(f) \qquad \qquad 0$$

$$0 \longrightarrow M \longrightarrow I_M \longrightarrow \Omega^{-1}(M) \longrightarrow 0$$

square Then we define a map $F:C_{lpha}
ightarrow I_{N}$ be the pushout property of the



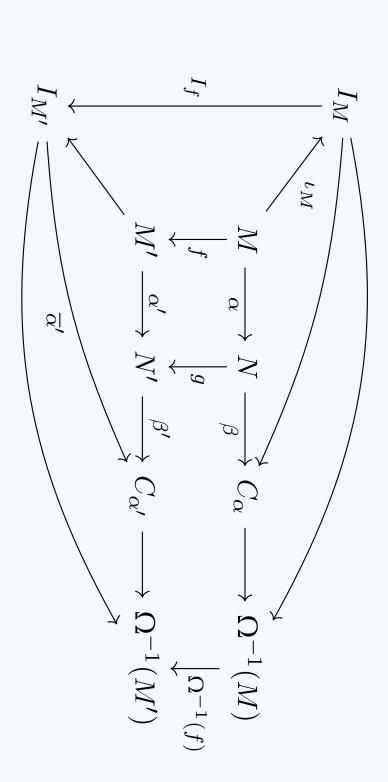
Then

$$N \to C_{\alpha} \xrightarrow{(F,-)} I_N \oplus \Omega^{-1}(M) \to \Omega^{-1}(N)$$

is a standard triangle as desired.



[T3] Situation:

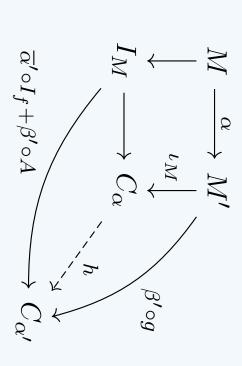




By assumption $g \circ \alpha - \alpha' \circ f : M \to N'$ factors through some projective i.e. through I_M , giving a map $A:I_M o N'$ satisfying

$$A \circ \iota_M = g \circ \alpha - \alpha' \circ f.$$

Then the pushout property of C_{α} gives us the right h:





[T4] Octahedron axiom. Start with three standard triangles

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} L' \xrightarrow{\gamma} \Omega^{-1}(M)$$

$$N \xrightarrow{\alpha'} L \xrightarrow{\beta'} M' \xrightarrow{\gamma'} \Omega^{-1}(N)$$

$$M \xrightarrow{\alpha''} L \xrightarrow{\beta''} N' \xrightarrow{\gamma''} \Omega^{-1}(M)$$

We need to construct another triangle

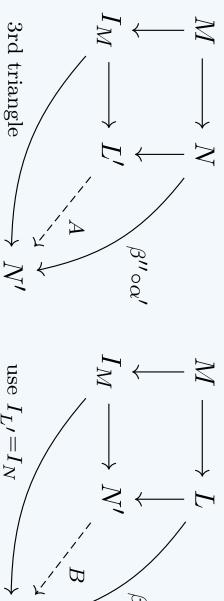
$$L' \xrightarrow{A} N' \xrightarrow{B} M' \xrightarrow{C} \Omega^{-1}(L')$$

where $C=\Omega^{-1}(eta)\circ\gamma'$ and show that it is distinguished and all occuring maps are compatible in the sense that they commute when arranged in an octahedron.



Proot

The pushout property of L^\prime gives us the map A



the commuting relations in the pushout setting on shows that A,B and $I_{L'}=I_N$. Then the pushout property of N' gives us the map B. Using the composition $N \hookrightarrow L' \hookrightarrow I_{L'}$ (still injective!), i.e. we identify Main clue: We modify the second triangle: Instead of $N \hookrightarrow I_N$ we use C satisfy all the required relations



It remains to show that $L' \xrightarrow{A} N' \xrightarrow{B} M' \xrightarrow{C} \Omega^{-1}(L')$ is distinguished. The identification $I_{L'} = I_N$ yields $\Omega^{-1}(N) = \Omega^{-1}(L')$. Then in the diagram

$$I_{L'} \longleftarrow N \longrightarrow L'$$

$$\downarrow M' \longleftarrow L \longrightarrow N'$$

$$B$$

showing that the above triangle is standard both squares are pushouts, hence the outer rectangle is also a pushout,



Remark (The stable category and group cohomology)

 $P_*=(\cdots \xrightarrow{\partial_2}P_1 \xrightarrow{\partial_1}P_0 \xrightarrow{\partial_0} M \to 0)$. Then applying $\operatorname{Hom}_{k[G]}(-,N)$ we For a $k[G]{\operatorname{\mathsf{-module}}}\ M$ choose a projective resolution

get a cochain complex

$$0 \to \operatorname{Hom}_{k[G]}(P_0, N) \to \operatorname{Hom}_{k[G]}(P_1, N) \to \cdots$$

Then $\operatorname{Ext}^n_{k[G]}(M,N)$ is defined as the n-cohomology group of this complex

$$H^n(G,M) := \operatorname{Ext}_{k[G]}^n(k,M).$$

We then have the following connection:

$$H^n(G, M) = \operatorname{Hom}_{k[G]}(\Omega^n(k), M).$$



Remark (The stable category and group cohomology)

For n = 1 this is

$$\operatorname{Ext}_{k[G]}^{1}(M,N) = \operatorname{Hom}_{k[G]}(\Omega(M),N).$$

 $\zeta' \circ \partial_1 : P_1 o N$ an element in $\operatorname{Ext}^1_{k[G]}(M,N)$. Similarly any map $\zeta':\Omega(M) o N$ represents via the composition it comes from ∂_1 , i.e. factors through $\ker \partial_0 = \Omega(M)$. $\operatorname{Ext}^{\scriptscriptstyle 1}_{k[G]}(M,N)$ come from cocycles $\zeta:P_0 o M$ where "cocycle" means that This already follows immediately from the definition! Elements in

their difference factors through a projective k[G]-module One then shows that two such maps ζ', ζ'' represent the same element iff

Thanks for listening!