Stable module categories

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Theorem (projective modules)

*Recall that an R-module P is **projective** if one of the following equivalent conditions holds:*

(i) *We have the following uniersal lifting property:*

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists f'"'] \arrow[d, "f"] & \\ M \arrow[r, two heads, "p"'] & N \arrow[r] & 0 \end{tikzcd}$$

(ii) *The functor* $\mathrm{Hom}_{\mathrm{Mod}(R)}(P, -)$ *is exact.*

(iii) *Any short exact sequence* $0 \to M \to N \to P \to 0$ *of* R*-modules splits.*

(iv) P *is a direct summand of a free* R*-module.*

Theorem (injective objects)

*Recall that an R-module I is **injective** if one of the following equivalent conditions holds:*

(i) *We have the following universal lifting property:*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, hook, "i"] \arrow[d, "f"'] & N \arrow[dl, dashed, "\exists f'"] \\ & I & \end{tikzcd}$$

(ii) *The functor $\mathrm{Hom}_{\mathrm{Mod}(R)}(-, I)$ is exact.*

(iii) *Any short exact sequence $0 \to I \to M \to N \to 0$ of R-modules splits.*

Definition (projective cover)

For an R-module M a *projective cover* is a pair (P, π) where P is projective and $\pi : P \to M$ is an epimorphism.Define

$$\Omega(M, P, \pi) = \ker(\pi).$$

Definition (injective hull)

For an R-module M an *injective hull* is a pair (I, ι) where I is injective and $\iota : M \to I$ is a monomorphism.Define

$$\Omega^{-1}(M, I, \iota) = \operatorname{coker}(\iota).$$

Theorem

(i) *The category* $\mathrm{Mod}(R)$ *has enough projectives in the sense that for any* R*-module* M *there is a projective cover.If* (P, π), (P', π') *are two projective covers then by Schanuel's lemma*

$$\ker(\pi) \oplus P' \cong \ker(\pi') \oplus P.$$

(ii) *The category* $\mathrm{Mod}(R)$ *has enough injectives in the sense that for any* R*-module* M *there is an injetive hull.If* (I, ι), $(', \iota')$ *are two injective hulls then similarly*

$$\mathrm{coker}(\iota) \oplus I' \cong \mathrm{coker}(\iota') \oplus I.$$

Definition (stable module category)

For R-modules M, N let

$$\mathrm{PHom}_{\mathrm{Mod}(R)}(M, N) = \left\{ f \in \mathrm{Hom}_{\mathrm{Mod}(R)}(M, N) \mid f : M \to P \to N \right\}$$

be the subspace of $\mathrm{Hom}_{\mathrm{Mod}(R)}(M, N)$ consisiting of morphisms which factor through a projective module.
Then we define the *projectively stable module category* $\mathrm{PStMod}(R)$ of R to be the category with

$$\mathrm{ob}(\mathrm{PStMod}(R)) = \mathrm{ob}(\mathrm{Mod}(R)),$$

$$\mathrm{Hom}_{\mathrm{PStMod}(R)}(M, N) = \mathrm{Hom}_{\mathrm{Mod}(R)}(M, N) / \mathrm{PHom}_{\mathrm{Mod}(R)}(M, N).$$

Remark

- Two R-modules M, N are isomorphic in $\mathrm{PStMod}(R)$ if and only if $M \oplus P \cong N \oplus Q$ for some projective modules P, Q.
  If $M \oplus P \cong N \cong Q$ via inverse isomorphisms f, g, then write

  $$f = \begin{pmatrix} f_{MN} & f_{PN} \\ f_{MQ} & f_{PQ} \end{pmatrix}, \qquad g = \begin{pmatrix} g_{NM} & g_{QM} \\ g_{NP} & g_{QP} \end{pmatrix}.$$

  Then the conditions

  $$g \circ f = \mathrm{id}_{M \oplus P} = \begin{pmatrix} \mathrm{id}_M & 0 \\ 0 & \mathrm{id}_P \end{pmatrix}, \qquad f \circ g = \mathrm{id}_{N \oplus Q} = \begin{pmatrix} \mathrm{id}_N & 0 \\ 0 & \mathrm{id}_Q \end{pmatrix}$$

  show $M \cong N$ via f_{MN} and g_{NM}. Still to shows: If $M \cong N$ there are projectives P, Q as above.

Remark

- The procedure also works with injectives instead of projectives (one then mods out $\mathrm{IHom}_{\mathrm{Mod}(R)}(M, N)$) and ends up with the *injectively stable module category* $\mathrm{IStMod}(R)$.
- There is a tensor product on the stabe categories inherited from $\mathrm{Mod}(R)$. This will be used later in the workshop

Definition (two functors)

In the projectively stable category we can define

$$\Omega(M) := \ker(\pi),$$

for some projective cover $\pi : P \to M$. In the injectively stable category we can define

$$\Omega^{-1}(M) := \operatorname{coker}(\iota)$$

for some injective hull $\iota : M \to I$.
Note that these definitions are independent of the choice of the projective cover π resp. the injective hull ι. We define moreover

$$\Omega^n(M) = \Omega(\Omega^{n-1}(M)), \qquad \Omega^{-n}(M) = \Omega^{-1}(\Omega^{-n+1}(M)).$$

Theorem (functorialty)

The assignment Ω *(*Ω^{-1}*) defines and endofunctor on* $\mathrm{PStMod}(R)$ *(*$\mathrm{IStMod}(R)$*).*

Proof

We exemplarily prove functoriality for Ω; the proof for Ω^{-1} works similarly.
Let $f \in \mathrm{Hom}_{\mathrm{PStMod}(R)}(M, N)$ and consider projective covers of M and N.

$$\begin{tikzcd}
\Omega(M) \arrow[r, hook] \arrow[d, dashed, "\Omega(f)"'] & P \arrow[r, two heads, "\alpha"] \arrow[d, dashed, "\gamma"'] & M \arrow[d, "f"] \\
\Omega(N) \arrow[r, hook] & Q \arrow[r, two heads, "\beta"] & N
\end{tikzcd}$$

Since β is an epimorphism we can lift $f \circ \alpha$ to a map $\gamma : P \to Q$. Define $\Omega(f) := \gamma|_{\Omega(M)}$.
This is well-defined: For two lifts γ_1, γ_2 the difference satisfies $\mathrm{im}(\gamma_1 - \gamma_2) \subset \ker \beta = \Omega(N)$, i.e. $\gamma_1 - \gamma_2$ factors through $\Omega(N)$, hence $\Omega(f_1) - \Omega(f_2)$ factors through P.

Definition

Let G be a finite group and k a field. Then

$$k[G] := \bigoplus_{g \in G} kg$$

is called the *group algebra of* G *over* k. If $H \subset G$ is a subgroup we have an inclusion $\iota : k[H] \hookrightarrow k[G]$. If further M is a $k[G]$-module, the *restriction of* M *to H* is the module M with multiplication $h \cdot m := \iota(h) \cdot m$ for $h \in k[H]$.

Example

1. If $R = \mathbb{Z}$ then $\mathrm{Mod}(R) = \mathrm{AbGrp}$ is the category of abelian groups. Then an R-module is projective iff it is free. Hence two finitely generated $\mathbb{Z}$-modules are isomorphic in $\mathrm{PStMod}(\mathbb{Z})$ iff their torsion part coincides.
2. If $R = k$ for a field then $\mathrm{Mod}(R) = \mathrm{Vect}_k$ is the category of k-vector spaces.Since any vector space is free hence projective in $\mathrm{PStMod}(k)$ all vector spaces are isomorphic to the trivial vector space.
3. Let $G = \mathbb{Z}/2$, k a field of characteristic 2 and $R = k[G] = k[x]/x^2$. One can show that for any R-module M we have

$$M \text{ is projective} \quad \Leftrightarrow \quad \text{for all } 0 \neq \alpha \in k \text{ the restriction } M|_{\langle 1+\alpha x\rangle} \text{ is free}$$

where $\langle 1 + \alpha x\rangle \subset k[x]/x^2$ is cyclic of order 2 (Dade, 1978).

Theorem

Let $R = k[G]$. Then any R-module is projective iff it is injective, i.e. the algebra$k[G]$ is Frobenius.

Proof

The Kronecker symbol $\delta_{g,h}$ can be extended $k[G]$-linearly to a bilinear, non-degenrate and G-invariant pairing $k[G] \times k[G] \to k$. This induces $k[G] \cong k[G]^*$ as k-vector spaces and by G-invariance even as $k[G]$-modules.
$\Rightarrow k[G]$ is *self-injective*, i.e. injective as a $k[G]$-module, since $k[G] \cong k[G]^*$ is free and thus projective.
direct sums of injectives are injective (Bass-Papp) $\Rightarrow$ free modules are injective.
P projective $\Rightarrow P$ is direct summand of a free, i.e. injective module $\Rightarrow P$ injective
Similarly I injective $\Rightarrow I^*$ is projective, i.e. direct summand of a free module.
$I \cong I^* \Rightarrow I$ is projective.

Definition

We write $\mathrm{StMod}(R) := \mathrm{PStMod}(R) = \mathrm{IStMod}(R)$ for the stable module category of a Frobenius category.

Corollary

The funcotrs Ω and Ω^{-1} on $\mathrm{StMod}(k[G])$ are inverse to each other. (Recall: $\Omega(M) = \ker(P \to M)$ and $\Omega^{-1}(M) = \mathrm{coker}(M \to I)$).

Proof

for a projective cover $\alpha : P \to M$ of some $k[G]$-module M we have an exact sequence

$$0 \to \Omega(M) \overset{\iota}{\hookrightarrow} P \overset{\alpha}{\to} M \to 0,$$

i.e. $M = \mathrm{coker}(\iota)$.
Since P is also injective, $\Omega^{-1}(\Omega(M)) = \mathrm{coker}(\iota) = M$.
Similarly $\Omega(\Omega^{-1}(M)) = M$.

Definition (triangles)

Let $\alpha \in \mathrm{Hom}_{\mathrm{StMod}(k[G])}(M, N)$. Then α induces a diagram

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "\iota_M"'] & N \arrow[d, "\beta"] \\
I_M \arrow[r, "\overline{\alpha}"'] \arrow[d, "\bar{\iota}_M"'] & C_\alpha \arrow[d, "\gamma"'] \\
\Omega^{-1}(M) \arrow[r, equal] & \Omega^{-1}(M)
\end{tikzcd}$$

where the upper square is a pushout and γ exists by the pushout property of C_α. The triangle $M \to N \to C_\alpha \to \Omega^{-1}(M)$ is called *standard*. We call a triangle *distinguished* if it is isomorphic to a standard triangle.

Theorem

$\mathrm{StMod}(k[G])$ *is a triangulated category with shift* Ω^{-1}*.*

Proof

We will more generally show that the stable category of any Frobenius category (where injective and projectives coincide) is triangulated.

[T1] By construction distinguished triangles are closed under isomorphisms and any morphism indues a distinguished triangle.
Moreover the identity $1_M : M \to M$ induces $C_{1_M} = I_M \cong 0$ in the stable category, i.e. $M \to M \to 0 \to \Omega^{-1}(M)$ is a standard triangle.

[T2] Given a distinguished triangle $M \to N \to L \to \Omega^{-1}(M)$ we need to show that $N \to L \to \Omega^{-1}(M) \to \Omega^{-1}(N)$ is distinguished.
Restrict to standard triangles, i.e. we have a standard triangle $M \xrightarrow{\alpha} N \to C_\alpha \to \Omega^{-1}(M)$.

Proof

Strategy: "Add" some injective I to $\Omega^{-1}(M)$ (which is an isomorphism in $\mathrm{StMod}(k[G])$) such that $\Omega^{-1}(M) \oplus I$ identifies with the pushout and the shifted triangle is even standard (hence distinguished). For $f : M \to N$ consider as before

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & I_N \arrow[r] & \Omega^{-1}(N) \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, "f"] & I_M \arrow[r] \arrow[u, dashed, "f'"'] & \Omega^{-1}(M) \arrow[r] \arrow[u, dashed, "\Omega^{-1}(f)"'] & 0
\end{tikzcd}$$

Then we define a map $F : C_\alpha \to I_N$ be the pushout property of the square

Proof

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d] & N \arrow[d] \arrow[ddr, bend left] & \\
I_M \arrow[r] \arrow[drr, bend right, "f'"'] & C_\alpha \arrow[dr, dashed, "F"] & \\
& & I_N
\end{tikzcd}$$

Then

$$N \to C_\alpha \xrightarrow{(F,-)} I_N \oplus \Omega^{-1}(M) \to \Omega^{-1}(N)$$

is a standard triangle as desired.

Proof

[T3] Situation:

$$
\begin{tikzcd}
I_M \arrow[dddd, "I_f"'] \arrow[ddrrr, bend left] \arrow[ddrrrr, bend left] & & & & \\
& & & & \\
& M \arrow[uul, "\iota_M"'] \arrow[r, "\alpha"'] \arrow[d, "f"] & N \arrow[r, "\beta"'] \arrow[d, "g"] & C_\alpha \arrow[r] & \Omega^{-1}(M) \arrow[d, "\Omega^{-1}(f)"] \\
& M' \arrow[ddl] \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & C_{\alpha'} \arrow[r] & \Omega^{-1}(M') \\
I_{M'} \arrow[urrr, bend right, "\overline{\alpha}'"] \arrow[urrrr, bend right] & & & &
\end{tikzcd}
$$

Proof

By assumption $g \circ \alpha - \alpha' \circ f : M \to N'$ factors through some projective, i.e. through I_M, giving a map $A : I_M \to N'$ satisfying

$$A \circ \iota_M = g \circ \alpha - \alpha' \circ f.$$

Then the pushout property of C_α gives us the right h:

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d] & M' \arrow[d, "\iota_M"'] \arrow[ddr, bend left, "\beta' \circ g"] & \\
I_M \arrow[r] \arrow[drr, bend right, "\overline{\alpha}' \circ I_f + \beta' \circ A"'] & C_\alpha \arrow[dr, dashed, "h"] & \\
& & C_{\alpha'}
\end{tikzcd}$$

Proof

[T4] Octahedron axiom. Start with three standard triangles

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} L' \xrightarrow{\gamma} \Omega^{-1}(M)$$

$$N \xrightarrow{\alpha'} L \xrightarrow{\beta'} M' \xrightarrow{\gamma'} \Omega^{-1}(N)$$

$$M \xrightarrow{\alpha''} L \xrightarrow{\beta''} N' \xrightarrow{\gamma''} \Omega^{-1}(M)$$

We need to construct another triangle

$$L' \xrightarrow{A} N' \xrightarrow{B} M' \xrightarrow{C} \Omega^{-1}(L')$$

where $C = \Omega^{-1}(\beta) \circ \gamma'$ and show that it is distinguished and all occuring maps are compatible in the sense that they commute when arranged in an octahedron.

Proof

The pushout property of L' gives us the map A

$$
\begin{tikzcd}
M \arrow[r] \arrow[d] & N \arrow[d] \arrow[ddr, bend left, "\beta''\circ\alpha'"] & \\
I_M \arrow[r] \arrow[drr, bend right, "\text{3rd triangle}"'] & L' \arrow[dr, dashed, "A"] & \\
 & & N'
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r] \arrow[d] & L \arrow[d] \arrow[ddr, bend left, "\beta"] & \\
I_M \arrow[r] \arrow[drr, bend right, "\text{use } I_{L'}=I_N"'] & N' \arrow[dr, dashed, "B"] & \\
 & & M'
\end{tikzcd}
$$

Main clue: We modify the second triangle: Instead of $N \hookrightarrow I_N$ we use the composition $N \hookrightarrow L' \hookrightarrow I_{L'}$ (still injective!), i.e. we identify $I_{L'} = I_N$. Then the pushout property of N' gives us the map B. Using the commuting relations in the pushout setting on shows that A, B and C satisfy all the required relations.

Proof

It remains to show that $L' \xrightarrow{A} N' \xrightarrow{B} M' \xrightarrow{C} \Omega^{-1}(L')$ is distinguished. The identification $I_{L'} = I_N$ yields $\Omega^{-1}(N) = \Omega^{-1}(L')$. Then in the diagram

$$\begin{tikzcd}
I_{L'} \arrow[d] & N \arrow[l] \arrow[r] \arrow[d] & L' \arrow[d, "A"] \arrow[ll, bend right] \\
M' & L \arrow[l] \arrow[r] & N' \arrow[ll, bend left, "B"]
\end{tikzcd}$$

both squares are pushouts, hence the outer rectangle is also a pushout, showing that the above triangle is standard.

Remark (The stable category and group cohomology)

For a $k[G]$-module M choose a projective resolution $P_* = (\cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0)$. Then applying $\mathrm{Hom}_{k[G]}(-, N)$ we get a cochain complex

$$0 \to \mathrm{Hom}_{k[G]}(P_0, N) \to \mathrm{Hom}_{k[G]}(P_1, N) \to \cdots$$

Then $\mathrm{Ext}^n_{k[G]}(M, N)$ is defined as the n-cohomology group of this complex and

$$H^n(G, M) := \mathrm{Ext}^n_{k[G]}(k, M).$$

We then have the following connection:

$$H^n(G, M) = \mathrm{Hom}_{k[G]}(\Omega^n(k), M).$$

Remark (The stable category and group cohomology)

For $n = 1$ this is

$$\mathrm{Ext}^1_{k[G]}(M, N) = \mathrm{Hom}_{k[G]}(\Omega(M), N).$$

This already follows immediately from the definition! Elements in $\mathrm{Ext}^1_{k[G]}(M, N)$ come from cocycles $\zeta : P_0 \to M$ where "cocycle" means that it comes frfom ∂_1, i.e. factors through $\ker \partial_0 = \Omega(M)$.
Similarly any map $\zeta' : \Omega(M) \to N$ represents via the composition $\zeta' \circ \partial_1 : P_1 \to N$ an element in $\mathrm{Ext}^1_{k[G]}(M, N)$.
One then shows that two such maps ζ', ζ'' represent the same element iff their difference factors through a projective $k[G]$-module.

Thanks for listening!