GRK 2240 WORKSHOP: C_i-FIELDS

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Introduction

The main subject of this workshop is to study fields in which the following phenomenon arises:

a homogeneous polynomial which has sufficiently more variables than its degree has a non trivial solution.

There are several ways in which one can try to formalize what "sufficiently" should mean. We will focus on a variant introduced by Lang in his thesis under the name of *C_i-fields*.

Definition 1 (Lang)**.** Let i be a non-negative integer. A field k is called C_i if every homogeneous polynomial with coefficients in k of degree d and in strictly more than d^i variables has a non-trivial solution in k.

It is an easy exercise to show that k is C_0 if and only if it is algebraically closed (do it!). Fields which are C_1 were studied by Tsen, and Artin called them *quasi-algebraically closed* before Lang introduced the C_i terminology. Early results imply the following connection to Brauer groups:

Lemma 2. *If k is C_1, then the Brauer group of k is trivial.*

In spite of the previous result and a result of Wedderburn which implies that the Brauer group of a finite field is trivial, Artin conjectured that finite fields are C_1. This was indeed later confirmed by Chevalley [2] and follows by the following stronger result of Warning [12] (nowadays called Chevalley-Warning):

Theorem 3 (Chevalley-Warning)**.** *Let f be a polynomial in n variables with coefficients in a finite field k and let d be its degree. If $n > d$, then the number of solutions of f in k is congruent to 0 modulo p. In particular, finite fields are C_1.*

Lang, Nagata and Tsen studied the behaviour of the C_i condition under field extensions.

Theorem 4 (Lang)**.** *Let k be a C_i field. Then every algebraic extension of k is C_i.*

Theorem 5 (Tsen/Lang-Nagata)**.** *Let k be a C_i-field. If K is an extension of k of transcendence degree n, then K is C_{i+n}.*

Later, Greenberg [5] showed a similar result for fields of power series.

Theorem 6 (Greenberg)**.** *Let k be a C_i field. Then $k((t))$ is C_{i+1}.*

As a Corollary one obtains that the field $\mathbb{F}_p((t))$ is C_2 (and also that $\mathbb{C}((t))$ is C_1 and hence, has trivial Brauer group). Given the similarity between $\mathbb{F}_p((t))$ and the field of p-adic numbers $\mathbb{Q}_p$, Artin conjectured that $\mathbb{Q}_p$ was also C_2. However, this time Artin was wrong. Although his conjecture was confirmed in degree $d \leqslant 3$ (see [9]), Terjanian [11] found counter-examples in degree 4. Surprisingly, despite this negative answer, the following theorem of Ax-Kochen [1] shows that Artin was almost correct in his guess.

Theorem 7 (Ax-Kochen)**.** *Fix $d > 0$. Then, there is a finite set X_d of prime numbers such that for every $p \notin X_d$, every homogenous polynomial with coefficients in $\mathbb{Q}_p$ with strictly more than d^2 variables has a non-trivial solution in $\mathbb{Q}_p$.*

Their proof uses some mild tools from model theory. One of the main tools which we will take as a black box is the following result of Ax and Kochen also proved independently by Ershov (all terms to be later defined during the workshop):

Theorem 8 (Ax-Kochen/Ersov)**.** *Let φ be a first-order sentence in the language of valued fields. Then there is a finite set of prime numbers E_φ such that for all $p \notin E_\varphi$*

$$\varphi \text{ holds in } \mathbb{Q}_p \;\Leftrightarrow\; \varphi \text{ holds in } \mathbb{F}_p(\!(t)\!).$$

An easy application of Theorem 8 yields the proof of Theorem 7.

The main objective of workshop is to prove all the above mentioned theorems. Most objects such as Brauer groups, valued fields and ultraproducts will be introduced from scratch. The main result which will be used as a black box is Theorem 8, although all the terms involved in it will be explained.

Most articles so far cited are very well written and interesting to read on their own and speakers are more than welcomed to read the original papers. There is a beautifully written survey book by M. Greenberg [6] on the subject which we can follow for all results except of Theorem 7. The master thesis of A. Kruckman [8] is also a very good survey of Ax-Kochen's theorem which also includes most results we will cover in the workshop. The following is a more detailed schedule for the talks together with other potential references.

Lecture 1: Overview of C_i-fields and motivation

This first lecture should introduce the audience to C_i-fields and provide an overview of the results we will tackle in the workshop. Besides discussing several examples the lecture should contain:

(1) an introduction to Brauer groups and its connexion to C_i-fields (i.e., a proof of Lemma 2);
(2) a proof that finite fields are C_1 via a proof of Chevalley-Warning (Theorem 3).

All this material is nicely explained and included in the Chapters 1 and 2 of [6] (they are quite short). Theorem 3 is quite classical and there are different variants, so feel free to show the one you find more interesting.

Lecture 2: Field extensions

This lecture will be devoted to study the behaviour of the C_i condition under field extensions. We will prove Theorems 4 and 5, which are contained in Chapter 3 of [6]. They also appear in [8] (but he also cites [6]).

Lecture 3: A result on $\mathbb{F}_p(\!(t)\!)$

This lecture should provide definitions and basic background results on valued fields and valuation rings, with a special treatment of complete discrete valuation rings. The lecturer should give a sketch of the proof of Theorem 6 for the particular case of the field $\mathbb{F}_p(\!(t)\!)$, and hence conclude that $\mathbb{F}_p(\!(t)\!)$ is C_2. This can be found in Chapter 4 of [6].

Lectures 4 and 5: Greenberg's theorem

These two lecture will be dedicated to show Theorem 6 in its full generality. This is done by showing a result about solutions of polynomials in discretely valued fields due to Greenberg in [5]. The proof uses some algebraic geometry and in order to be as much as self-contained as possible, it is splited in two lectures. A discussion about related problems (which are also discussed in [5]) would be interesting if time permits. If two different lecturers are assigned for Lectures 4 and 5, they should communicate in order to split the content of the proof.

Lecture 6: Negative results on $\mathbb{Q}_p$

In this lecture concerns counter-examples to Artin's conjecture and related literature. It should present the first counter-examples due to Terjanian [11] (hopefully together with a sketch of proof) and some indications/comments about positive answers in lower degree (see [9]). If time allows, the lecturer could also present more general results showing that $\mathbb{Q}_p$ is not C_i for any $i \geqslant 1$!

Lecture 7: The transfer principle and Ax-Kochen's theorem

In this lecture deals with Theorems 7 and 8. On the one hand, the lecturer should give some simple examples providing some intuition of the transfer principle stated in Theorem 8 (you can ask either to Prof. Halupczok or to me for some ideas on such examples). On the other hand, the lecturer should provide the needed background definitions to formally understand every term involved in the theorem (i.e., first order sentence, language of valued fields, etc.). No proof of Theorem 8 will be given but just an intuitive account. Once establishing such an intuition, the lecturer should briefly explain how to formally deduce Theorem 7.

Lectures 8 and 9: Ultraproducts and transfer principles

The final two lectures are somewhat independent of the first 7 lectures, although their content is still related to Theorem 8. The purpose of these lectures is to introduce the students to the ultraproduct construction and show its relation to Theorem 8. Independently of C_i-fields, these lectures might be interesting for any algebraist.

Lecture 8: Ultraproducts and transfer principles I. In this lecture the ultraproduct construction will be introduced (say for algebraic structures such as groups, rings and fields). The speaker should try to provide as many examples and intuition as possible. In addition, he/she should discuss one of its fundamental results called Łoś theorem. For an algebraic intuition, see [10], and for formal definitions see [7, 4, 3]. The material is also discussed in [8].

Lecture 9: Ultraproducts and transfer principles II. In this lecture the relation between Theorem 8 and ultraproducts should be sketched, which is presented in [8]. Other transfer principles could also be discussed. One possibility is as first-order version of a Lefschetz transfer principle for algebraically closed fields. (see for instance [4, Theorem 4.1.4]).

References

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