GRK 2240 WORKSHOP: TENSOR TRIANGULAR GEOMETRY SUMMER SEMESTER 2021

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Introduction

This workshop is built around the topic of triangulated categories. The definition is usually credited to Jean-Louis Verdier in 1963 [14], but Dieter Puppe had already given a partial definition in 1962 and deserves honorable mention. Intuitively, a triangulated category can be thought of as *a category which behaves well with respect to homology.*

For Verdier, the most important application of the theory was the *derived category* of a ring or, more generally, a scheme. The derived category of a commutative ring R is the triangulated category of chain complexes of R-modules "up to quasi-isomorphism". One of the goals of the workshop is to make the construction of the derived category mathematically precise. This will require a fairly long sequence of definitions, but most of these definitions have other uses, and if they don't we try not to spend too much time on them. Derived categories are a very important object of study in many areas of modern mathematics, such as representation theory, algebraic topology, differential geometry, and algebraic geometry.

In recent years, Paul Balmer has taken triangulated methods into an exciting new direction [2] by taking into account the tensor product that exists on almost every triangulated category that appears in the wild. One of the surprising results of this theory is [2, theorem 54].

Theorem 1 (Balmer)**.** *Let X be a quasi-compact and quasi-separated scheme. Then there is an isomorphism* $\mathrm{Spec}(D^{\mathrm{perf}}(X)) \cong X$ *of ringed spaces.*

Here, $\mathrm{Spec}(D^{\mathrm{perf}}(X))$ is the *Balmer spectrum* of the tensor triangulated category of perfect complexes of $\mathcal{O}_X$-modules, a construction which will be explained in the course of the program, and which is analogous to the spectrum of a commutative ring. By contrast, an example of Mukai [9] shows that there are non-isomorphic schemes whose derived categories are equivalent as triangulated categories. For other interesting examples of this phenomenon, see [6].

The second half of the program will be an introduction to the theory of tensor triangular geometry and its applications, leading up to a sketch of the proof of theorem 1.

In summary, the goals of the program are to

(i) understand fundamental examples of abelian categories, categories of chain complexes and resolutions of objects;
(ii) understand key constructions in chain complexes such as the mapping cone;
(iii) import techniques and intuition from topology into homological algebra;
(iv) understand the derived category of a ring or scheme;
(v) gain some familiarity with the stable module category;
(vi) understand how the derived category and the stable module category are examples of tensor triangulated categories; and
(vii) be able to state the definition of the Balmer spectrum along with the main results regarding it.

The descriptions of the talks are meant to be guidelines for the lecturers, and they leave some room for their personal taste. The lists found in the description of each talk are the anchor points that keep the talks connected, but the order in which these items are presented is flexible.

Talk 1: Abelian categories

The goal of this talk is to make the audience familiar with the definition of an additive category and that of an abelian category. It should include some basic examples such as the category **Ab** of abelian groups, to pave the way for the various constructions in chain complexes. A good reference for abelian categories is [15, appendix A.4].

The lecturer should

(i) define additive categories;
(ii) show that an additive category with one object is a ring;
(iii) explain kernels and cokernels categorically in terms of pullbacks and pushouts;
(iv) define images and coimages in terms of kernels and cokernels;
(v) define abelian categories and (left or right) exact functors between them;
(vi) define projective and injective objects;
(vii) define projective [15, section 2.2] and injective [15, p. 2.3] resolutions, provided the abelian category has enough projectives or injectives;
(viii) give concrete examples of these constructions; and
(ix) state Freyd's embedding theorem [15, theorem 1.6.1].

For each standard example of an abelian category, mention that there is a tensor product on the category as well. State how the category of quasi-coherent sheaves on an affine scheme $\operatorname{Spec} A$ is equivalent to the category of A-modules, and mention that the category of quasi-coherent sheaves on a scheme is abelian. Try to make it as intuitive as possible for audience members who are not as familiar with schemes and sheaves. A good reference for categories of quasi-coherent sheaves on schemes is [13, section V.13.4].

Talk 2: Chain complexes

The second talk is devoted to categories of chain complexes, which play a crucial role in the construction of the derived category. Please mind the correct implementation of signs, they matter in this talk.

The lecturer should

(i) define the category of chain complexes of abelian groups and extend the definition to any abelian category (and mention cochain complexes);
(ii) define the homology of a chain complex and show that this construction is functorial;
(iii) define quasi-isomorphisms;
(iv) define the tensor product chain complex and the mapping chain complex of two chain complexes;
(v) define chain homotopies in the standard way and in the "topological" way;
(vi) show that chain homotopic maps induce the same maps in homology;
(vii) compare homotopy equivalences and quasi-isomorphisms;
(viii) define the shift of a chain complexes by tensoring with $\mathbb{Z}[n]$ and showing that this is the usual shift; and
(ix) define the category of bounded complexes of finitely generated modules.

It is probably preferable to do all of this in the category of chain complexes of abelian groups, and then explain how to generalize it to arbitrary abelian categories.

Regarding item (v), follow [7, section 6] for the construction of the path object and the topological definition of chain homotopy, but with the chain complex $C = \mathbb{Z}$, which is the unit for the tensor product. Then show that for an arbitrary chain complex C, we can just tensor C with the construction for $\mathbb{Z}$.

Feel free to consult and sample from [5, section 7], which is a great reference for the topological perspective. For a more basic introduction, see [15, sections 1.1-1.5], especially the topological remark in section 1.5, or [1, section 3].

Talk 3: Triangulated categories and localization

This talk introduces the triangulated categories and their localizations. Most importantly, it should contain a rigorous definition of a triangulated category. Feel free to omit the precise definition of multiplicative systems and focus on the existence and universal property of localizations.

The lecturer should:

(i) give the axioms of a triangulated category;
(ii) provide some intuition for the octahedral axiom;
(iii) define exact functors between triangulated categories as well as homological functors;
(iv) prove some basic results about triangulated categories to acquaint the audience to their particular yoga;
(v) show that the category $\mathrm{Ch}(\mathbf{Ab})$ of chain complexes is *not* triangulated;
(vi) show that the category $K(\mathbf{Ab})$ of chain complexes up to chain homotopy *is* triangulated;
(vii) define thick subcategories of a triangulated category, and note that they are the same as saturated and épaisse subcategories;
(viii) define the localization of a triangulated category with respect to a class of objects or morphisms as a triangulated category with a universal property; and
(ix) define the derived category $D(\mathbb{Z})$ as the localization of $K(\mathbf{Ab})$ with respect to the quasi-isomorphisms.

Important examples of triangulated categories will follow in the next talk, so this talk should really focus on the yoga of triangulated categories [15, section 10.2]. Also, the set-theoretic problems with localizations should be swept under the rug.

Talk 4: Derived categories and derived functors

This talk focuses on one of the main examples of a triangulated category: the derived category of an abelian category.

The lecturer should

(i) restate the definition of the derived category of an abelian category, and the various bounded versions thereof;
(ii) explain why there are two localizations in the definition, and provide some historical background;
(iii) give at least one explicit example of the derived category of a scheme, one obvious candidate being the derived category of $\mathbb{P}^n$;
(iv) discuss derived functors following [11, section 6], including sheaf cohomology as the derived functor of global sections, Tor-groups and Ext-groups; and
(v) compare derived functors to more *ad hoc* definitions of (co)homology.

When discussing derived functors, the focus should be on the elegance of the derived functors as exact functors, rather than on the technical details. Feel free to include more examples of derived categories.

Talk 5: Stable module categories

The last talk of the first half of the program takes a closer look at the stable module category of a group ring $k[G]$, which is a triangulated category with G-representations as its objects. The main reference for this talk is [4, section 1.1].

The lecturer should

(i) define the stable module category of a ring R;
(ii) define the operations Ω and Ω^{-1} on the stable module category and give some concrete examples;
(iii) mention that the stable module category has a tensor product;
(iv) show that if $R = k[G]$ for some group G, an R-module is projective if and only if it is injective and deduce that Ω and Ω^{-1} are inverse to each other;

(v) sketch how the stable module category of $k[G]$ is triangulated, including the cone construction; and
(vi) relate group cohomology to morphisms in the stable module category [4, section 1.2.2].

Feel free to mention that $\mathbf{Mod}(k[G])$ is a Frobenius category [4, sections 1.3.4, 1.3.5], and that the stable module category of $k[G]$ measures how far $k[G]$ is from being semi-simple. Time permitting, it might be interesting to include a discussion of the variety of a G-representation [10, section 9.3].

Talk 6: Summary

In this talk, the lecturer should summarize what has been defined and discussed up until this point in the program. The focus should be on the derived category and the stable module category, and the (co)homological methods that come with them. Other than that, the lecturer should feel free to interpret this talk as they like. One idea is to give more concrete examples of derived categories, stable module categories, sheaf cohomology or group cohomology, and summarize the previous talks using such examples.

Talk 7: Tensor triangulated categories

The first talk of the second half of the program introduces tensor triangulated categories, whose shadow we have already seen in all of the examples of the first half of the program. From now on, the main reference is [2].

The lecturer should

(i) define tensor triangulated categories [2, definition 3];
(ii) give the example of the derived category $D^{\mathrm{perf}}(X)$ of perfect complexes over a scheme X and give a simpler description for quasi-projective and affine schemes;
(iii) define compact objects [10, p. 1.3.9] and prove that these correspond to perfect complexes [10, theorem 2.2];
(iv) define support data, ideals, radical ideals and prime ideals of tensor triangulated categories [2, theorem 6, definition 7, construction 8]; and
(v) define the spectrum of a tensor triangulated category, including its structure sheaf [2, constructions 24, 29].

For this talk, it might also be useful to consult [3] and [4, sections 2.1.2, 2.1.4]. Carefully address the analogy between spectra of rings and spectra of tensor triangulated categories.

Talk 8: Thomason's classification of thick ideals

The last two talks are dedicated to a sketch of the proof of theorem 1. The key ingredients of this proof are [12, theorem 3.15] and [3, theorems 3.2, 5.2]. This talk should provide a proof of [12, theorem 3.15]. This result relies heavily on advanced methods in algebraic geometry, which cannot be covered in this program, so please treat [12, lemma 3.3] and similar results as a black box, but provide some geometric intuition for them. The lecturer should also state how this result simplifies for $\operatorname{Spec} A$, where A is a noetherian ring.

Talk 9: The Balmer spectrum recovers quasi-compact quasi-separated schemes

This talk finishes the proof of theorem 1, by proving [3, theorems 3.2, 5.2]. If time permits, feel free to state the similar [2, theorem 57] and remark upon the striking unification of modular representation theory and algebraic geometry.

Bonus talk: Gluing Balmer spectra

This optional talk is about the gluing of Balmer spectra, and the similarities and differences between this method and the gluing of spectra of rings. One of the applications of this method allows one to define the Picard group of a Balmer spectrum, at the cost of a few caveats. The lecturer should

(i) review the gluing construction for spectra of rings;
(ii) explain [2, theorem 31], which states that objects in a tensor triangulated category with disconnected support decompose accordingly as direct sums;
(iii) state and explain the gluing construction [2, theorem 33] and its shortcomings due to non-uniqueness [2, remark 34];
(iv) define the Picard group of a Balmer spectrum [2, definition 35]; and
(v) explain the long exact Mayer Vietoris sequence for the Picard group [2, theorem 37], comparing it to the usual Mayer-Vietoris sequence for the Picard group of e.g. [8].

Take ample time for the review of the gluing construction for spectra of rings, since this method is crucial in scheme theory. Also take plenty of time for items (ii) and (iii), frequently returning to the classical gluing construction for intuition and explanation of pitfalls.

References

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