Workshop of the GRK2240

WINTERSEMESTER 2018/2019

Topic: stacks

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"Now the real clue here, I contend, is that we must not ask for the cohomology or the Picard group simply of a variety; there is a much better object, which is much more intrinsically related to the moduli problem ... "

- David Mumford, [5]

1 Introduction

1.1 Motivation

In the above quote, Mumford is talking about a *stack*, even though the term won't be coined for another four years, until he and Pierre Deligne write [1] together. Although nothing in the definition of a stack is intrinsic to algebraic geometry, the concept was born and nurtured within that area of mathematics by the likes of Alexander Grothendieck, the aforementioned David Mumford and Pierre Deligne, and Michael Artin. The first hints of their existence appeared in the study of algebro-geometric moduli problems, and it is in this context that we will study them. The categorical machinery required for their definition may be technical and difficult; the indemnification is that the language of stacks is clean, concise and powerful, as was first demonstrated in [1]. It gets rid of a plethora of special cases and artificial conditions that have to be imposed on various interesting moduli problems to make sense of them otherwise. Besides, it seems that stacks are finding more and more applications outside of the theory of moduli problems, so I think it is a good time for young algebraic geometers to learn about them.

The aim of this seminar is to gain sufficient familiarity with the topic to be able to formulate and think about our favorite moduli problems in the language of stacks, and to be able to read articles using this language.

1.2 Program outline

In the quoted article [5], Mumford admits that it is largely an exercise in coming to terms with Alexander Grothendieck's revolutionary generalized notion of topologies. As such, it provides interesting insight into the way people started thinking about stacks. The probably ambitious goal of this seminar is to understand the calculation of the Picard group of the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves as put forward in [3], which can be seen as an extension of Mumford's [5], and exhibits the maturation of the theory over the years. Hopefully, our understanding of stacks will mature in a similar way as we progress through the talks.

Any scheme X can be thought of as its corresponding *functor of points* $X : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$, assigning to a scheme S the set of morphisms $S \to X$. A morphism $f : S \to T$ of schemes induces a map $X(T) \to X(S)$ simply by precomposing any morphism $T \to X$ with f. In fact, this is an example of the famous *Yoneda embedding* and one of the most widely used results in category theory states that it is fully faithful. Thus the functor of points of X really contains all the information of X itself. Conversely, given a functor $F : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$, specifically one that encodes a classification problem (e.g. $S \mapsto \{\text{isomorphism classes of elliptic curves over } S\}$) we may wonder whether it is *represented* by an actual scheme X. In general, this will not be the case, but it turns out that by passing from schemes to a larger category, it can often be represented by something called a *stack*. A motivational introduction to stacks is [Stacks, Tag 072H].

The main reference for this seminar will be the *Stacks project* [Stacks], which is a vast encyclopedia of algebro-geometric knowledge. It will be navigated with the goal of approaching moduli problems in terms of stacks. The main result will be the construction of the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves and hopefully also the calculation of its Picard group, as mentioned. Additional references include Olsson's comprehensive *Algebraic Spaces and Stacks* ([6]) and Vistoli's *Notes on Grothendieck topologies, fibered categories and descent theory* ([7]), see the sections of the individual talks for more accurate and extensive references.

2 Talks

2.1 Fine and coarse moduli spaces

References: [Stacks, Tag 01J5], [Stacks, Tag 001L] and [4, chapter 2].

The aim of this talk is to familiarize ourselves with moduli problems and some of the basic approaches in tackling them.

Discuss the *functor of points of* a scheme S (c.f. [Stacks, Tag 01J5]). Follow [Stacks, Tag 001L] for the famous *Yoneda lemma* and the embedding $\mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ of a category in its category of presheaves. State the definition of a *moduli problem* [4, definition 2.8] and a *fine moduli space* [4, definition 2.10]. An example of a fairly simple moduli problem that is already somewhat interesting and admits a fine moduli space is [4, example 2.13]). See also [4, example 2.19].

Moduli problems rarely admit fine moduli spaces because of automorphisms of the objects. Illustrate this situation by giving a few examples. Give the definition of a *coarse moduli space* ([4, definition 2.24]). Discuss the three basic different strategies for constructing moduli spaces:

(i) modify the moduli problem so that it admits a fine moduli space;
(ii) forget about the automorphisms and try to find a *coarse moduli space*; and
(iii) try to find a *moduli stack* - this is the focus of this seminar.

Work out the example of the coarse moduli space of elliptic curves and show that it is not a fine moduli space.

2.2 Grothendieck topologies

References: [Stacks, Tag 00VG], [Stacks, Tag 00YW], [Stacks, Tag 00XZ], [7, section 2.3.1] and [7, section 2.3.2].

A topology on a set X can be seen as a subcategory (subject to some conditions) of the power set $\mathcal{P}(X)$ of X, which, when seen as a category, has the subsets of X as objects and inclusions of subsets as morphisms. Similarly, one can define the notion of a *topology on a category.* This idea was introduced by Grothendieck and was an essential ingredient in the development of the theory of stacks. The aim of this talk is to give the definition of and to familiarize ourselves with this more general notion.

Give the definition of a *Grothendieck topology* on a category $\mathcal{C}$, as in [Stacks, Tag 00VG] and [7, section 2.3.1]. Give the definition of a *topology* ([Stacks, Tag 00YW]) on a category $\mathcal{C}$ and elaborate on the fact that equivalent Grothendieck topologies give rise to the same topology. Show that the definition of a topology on a power set $\mathcal{P}(X)$ (regarded as a category) recovers the notion of a topology on X. If there is time, discuss the concept of a *localization of a site* (c.f. [Stacks, Tag 00XZ]).

2.3 Examples of Grothendieck topologies

References: , [Stacks, Tag 020K], [Stacks, Tag 021L], [7, section 2.3.1] and [7, section 2.3.2].

The purpose of this talk is to get our hands dirty with a few explicit examples of Grothendieck topologies. Feel free to add any examples not listed that could be relevant.

Work out some of the examples in [7] and [Stacks, Tag 020K], notably the fppf-topology ([Stacks, Tag 021L]), the fpqc-topology ([7, section 2.3.2]), the Zariski topology and the étale topology on Sch.

2.4 Fibered categories

References: [Stacks, Tag 02XJ], [Stacks, Tag 003S] and [7, section 3.3]

Let S be a scheme. Then the quasi-coherent sheaves on S form an abelian category $\mathrm{QCoh}(S)$. One can pull back a quasi-coherent sheaf on S along a morphism of schemes $f : X \to S$ to obtain a quasi-coherent sheaf on X. This defines a functor $f^* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(X)$. It turns out that there exists a category $\mathcal{QC}oh$ together with a functor $\mathcal{QC}oh \to \mathrm{Sch}$, which is actually a *category fibered in groupoids*. The goal of this talk is to define a formal framework for such things.

Using [Stacks, Tag 02XJ], give the definition of a fibered category and a few useful lemmas. Define morphisms of fibered categories. Perhaps a brief informal discussion of the 2-category of fibered categories is warranted. Move on to [Stacks, Tag 003S] about categories fibered in groupoids. Carefully consider a few examples, such as [Stacks, Tag 003U] and [Stacks, Tag 02C4].

2.5 Sheaves on sites

References: [Stacks, Tag 00VL], [Stacks, Tag 00W1], [7, section 2.3.3] and [7, section 2.3.6].

Let X be a topological space and $\mathcal{T}$ the set of opens of X, partially ordered by inclusion. Then $\mathcal{T}$ can be regarded as a category. A *sheaf (of sets) on* X is a functor $\mathcal{F} : \mathcal{T}^{\mathrm{op}} \to \mathrm{Set}$ which satisfies the following gluing property: given an open cover $\mathcal{U} = \{U_i \to U : i \in I\}$ of $U \in \mathcal{T}$ and sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \text{for all } i, j \in I,$$

there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$. The purpose of this talk is to generalize the notion of a sheaf on a topological space to that of a *sheaf on a site*.

Let $\mathcal{C}$ be a site and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. Follow [Stacks, Tag 00VL] for the definitions of a *separated* presheaf and a *sheaf on* $\mathcal{C}$. Highlight the representable functors of Sch/S, as in [7, theorem 2.55]. Discuss the concept of *sheafification* (c.f. [Stacks, Tag 00W1] and [7, section 2.3.7]). Following [Stacks, Tag 02W4], show that a descent datum corresponds to a sheaf that is "locally representable". For a site $\mathcal{C}$, show that there is a fibered category $\mathrm{Sh}/\mathcal{C} \to \mathcal{C}$, of which the fiber of $X \in \mathcal{C}$ is the sheaf category $\mathrm{Sh}(X)$.

2.6 Descent

References: [Stacks, Tag 02ZC], [Stacks, Tag 023U] and [7, section 4.1.2]

A useful concept in topology is that of gluing functions; descent generalizes this notion.

For a fibered category $p : \mathcal{S} \to \mathcal{C}$, an object U of $\mathcal{C}$ and a family $\mathcal{U}$ of morphisms in $\mathcal{C}$ with target U, define $\mathrm{DD}(\mathcal{U})$, the *category of descent data relative to* $\mathcal{U}$ (c.f. [Stacks, Tag 02ZC]) and show that this notion generalizes that of gluing. Give the definition of an *effective* descent datum. Following [7, section 4.1.2], relate various perspectives on descent data. Take a look at [Stacks, Tag 023F] about descent data for modules and [Stacks, Tag 023R] about fpqc descent. Consider some local properties of schemes for various topologies as in [Stacks, Tag 0347].

2.7 Examples of descent

References: [Stacks, Tag 0CDQ], [7, section 4.2] and [7, section 4.3.1]

The aim of this talk is to look at a few explicit examples of descent. Feel free to add more examples that could be illuminating.

First, consider the prototypical example of descent of vector bundles. Then consider the example of Galois descent as described in [Stacks, Tag 0CDQ]. Consider other examples of descent as described in [7, section 4.2] and [7, section 4.3.1].

2.8 Algebraic stacks

References: [Stacks, Tag 0268], [Stacks, Tag 02ZH], [Stacks, Tag 042Y], [Stacks, Tag 036X], [Stacks, Tag 02ZM], [Stacks, Tag 026N], [2], [7, section 4.1.3] and [7, section 4.1.6]

Let $\mathcal{C}$ be a site. Proposition 4.9 in [7] states that a presheaf $\mathcal{F}$ on $\mathcal{C}$ is a *stack* if and only if it is a sheaf. More generally, 'a stack is, morally, a sheaf of categories over $\mathcal{C}$', as Vistoli puts it in the beginning of [7, section 4.1.2].

Define *stacks* as in [Stacks, Tag 0268]. Discuss corollary 4.13 and 4.14 in [7]. Define *substacks*. Consider the *inertia stack* of a stack (c.f. [Stacks, Tag 036X]. Mention the analogue of sheafification, that is, *stackification* as discussed in [Stacks, Tag 02ZM]. Discuss the (historical) difference between *Deligne-Mumford stacks*, *algebraic stacks* and *Artin stacks*, following the conventions of [Stacks, Tag 026N]. Define properties of morphisms of algebraic stacks as in [2, definition 2.5].

2.9 Examples of stacks and their properties

References: [Stacks, Tag 036Z], [2] and [7, section 3.2]

Reconsider the examples from the first talk 2.1 as stacks. Work out an explicit example of the classifying stack $\mathcal{B}G$ of an algebraic group G, imitating [7, example 3.18] and elaborating on [Stacks, Tag 036Z]. The page [Stacks, Tag 02WE] lists many properties of morphisms of schemes that are stable under arbitrary base change and local on the base, so they also define properties of morphisms of stacks. Consider the valuative criterions [2, theorem 2.2 and theorem 2.3] for separation and properness.

2.10 The moduli stack of elliptic curves

References: [Stacks, Tag 072H], [1] and [2]

Define the *moduli stack* $\mathcal{M}_{1,1}$ *of elliptic curves*. Show that it is a Deligne-Mumford stack and explicitly give its atlas. Examine the properties of $\mathcal{M}_{1,1}$ and the morphism $\mathcal{M}_{1,1} \to \mathbb{A}^1$ to its coarse moduli space. It could be nice to more or less follow [Stacks, Tag 072H].

2.11 The Picard group of a moduli stack

References: [3], [5]

The aim of this talk is to classify line bundles on the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves, that is, to compute its Picard group. Follow parts of [5] and [3], and compare and contrast the two approaches.

References

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