Derived categories and derived functors

Recall:

$\mathcal{A}$ abelian cat., e.g. $\mathcal{A} = k\text{-vs}, R\text{-Mod}, Sh_X, Coh_X, QCoh_X$

$\rightsquigarrow Ch(\mathcal{A})$ = cat. of cochain complexes in $\mathcal{A}$

$\rightsquigarrow K(\mathcal{A})$ = cat. of cochain complexes in $\mathcal{A}$ up to cochain homotopy
:= loc. of $Ch(\mathcal{A})$ wrt homotopy equiv.

- Had cohomological functors
  $H^n : K(\mathcal{A}) \longrightarrow \mathcal{A}$, $A^\bullet \rightsquigarrow \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$, $n \in \mathbb{Z}$
- morph f in $K(\mathcal{A})$ or $Ch(\mathcal{A})$ is <u>quasi-isom.</u>
  $:\Leftrightarrow \forall n \in \mathbb{Z} : H^n(f)$ is isom.

$\rightsquigarrow D(\mathcal{A})$ = derived cat. of $\mathcal{A}$
:= loc. of $K(\mathcal{A})$ wrt quasi-isom.

We constructed:

- $\operatorname{ob} D(\mathcal{A}) := \operatorname{ob} Ch(\mathcal{A})$
- $\operatorname{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet) := \left\{ A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{f} B^\bullet \;\middle|\; \begin{array}{l} C^\bullet \in K(\mathcal{A}) \\ f \text{ morph} \\ s \text{ q-isom.} \end{array} \Big\} \text{ in } K(\mathcal{A}) \right\}$

  $f \circ C^\bullet \circ s$

Moreover:

- $D(\mathcal{A})$ is triang., $K(\mathcal{A}) \longrightarrow D(\mathcal{A})$ is triang.
  $H^n : D(\mathcal{A}) \longrightarrow \mathcal{A}$ cohom. functor
- Full triang. subcats. $D^-(\mathcal{A}), D^+(\mathcal{A}), D^b(\mathcal{A}) \subset D(\mathcal{A})$.

Just one loc. $Ch(\mathcal{A}) \longrightarrow D(\mathcal{A}) \rightsquigarrow$ get zigzags $A^\bullet \leftarrow C_1^\bullet \rightarrow C_2^\bullet \leftarrow \cdots \rightarrow B^\bullet$
of arbitr. lengths

Verdier : $Ch(\mathcal{A}) \longrightarrow K(\mathcal{A}) \longrightarrow D(\mathcal{A})$

Examples:

(1) $\mathcal{A} = (k\text{-vs})$. Let $V^\bullet \in D^b(\mathcal{A})$

Know: $V^n \cong \operatorname{Ker} d^n \oplus U^n \cong \operatorname{Im} d^{n-1} \oplus \underbrace{\operatorname{Ker} d^n / \operatorname{Im} d^{n-1}}_{= H^n(V^\bullet)} \oplus U^n$, for some $U^n \subset V^n$ s.t. $U^n \xrightarrow[\sim]{d^n} \operatorname{Im} d^n$

$\rightsquigarrow$ Get $pr: V^\bullet \twoheadrightarrow (H^\bullet(V))$, $\iota: (H^\bullet(V)) \hookrightarrow V^\bullet$ in $Ch(\mathcal{A})$

(with diff $\equiv 0$)

with $pr \circ \iota = id_{(H^\bullet(V))}$ and $\iota \circ pr \sim id_{V^\bullet}$.

$\Rightarrow V^\bullet$ and $(H^\bullet(V))$ homotopy equiv.

$\Rightarrow$ In $K^b(\mathcal{A})$ or $D^b(\mathcal{A})$: $V^\bullet \cong \bigoplus_{n \in \mathbb{Z}} H^n(V^\bullet)[n]$ ($H^n(V^\bullet)$ considered as complex concentrated in deg 0)

Upshot: $K^b(\mathcal{A}) \cong D^b(\mathcal{A}) \cong$ cat. of $\mathbb{Z}$-graded k-vs with fin. many non-zero components

$$V^\bullet \mapsto \bigoplus_{n \in \mathbb{Z}} H^n(V^\bullet)$$

(2) $\mathcal{A} = Coh_{\mathbb{P}^n_k} \rightsquigarrow D^b(\mathbb{P}^n_k) := D^b(\mathcal{A})$

Let $\Lambda := \Lambda k^{n+1}$ ext. alg., $\deg \xi = 1$, for $\xi \in k^{n+1} \setminus \{0\}$.

$M(\Lambda) :=$ cat. of fin. gen. graded Λ-mod.

$\cup$

$\mathcal{P} :=$ cat. of free —— " ——.

$\Phi: M(\Lambda) \longrightarrow Ch^b(\mathbb{P}^n_k) \longrightarrow D^b(\mathbb{P}^n_k)$

$\bigoplus_{j \in \mathbb{Z}} V_j \mapsto \mathcal{L}^\bullet$, $\mathcal{L}^j := V_j \otimes_k \mathcal{O}(j)$

(V_j ← graded part of deg j)

$d = \sum_{i=1}^{n+1} \xi_i \otimes x_i : \mathcal{L}^j \to \mathcal{L}^{j+1}$

ξ_i basis k^{n+1}, x_i dual basis

$\rightsquigarrow \xi_i : V_j \to V_{j+1}$, $v \mapsto \xi_i \cdot v$

$\rightsquigarrow x_i : \mathcal{O}(j) \to \mathcal{O}(j+1)$, $f \mapsto x_i \cdot f$ on sections

Thm: (Bernstein-Gelfand-Gelfand)
Φ induces an equiv. of cat.

$$M(\Lambda)/\mathcal{P} \xrightarrow{\sim} D^b(\mathbb{P}^n_k)$$

$f: V \to V'$ in $M(\Lambda)$ $\mathcal{P}$-equiv. to 0 $:\Leftrightarrow \exists P$ (with f factoring through P), $P \in \mathcal{P}$

$ob\, M(\Lambda)/\mathcal{P} = ob\, M(\Lambda)$, $\operatorname{Hom}_{M(\Lambda)/\mathcal{P}}(\cdot,\cdot) = \operatorname{Hom}_{M(\Lambda)}(\cdot,\cdot) / \mathcal{P}\text{-equiv to } 0$

Derived functors

Let $F: \mathcal{A} \to \mathcal{B}$ be an add. functor of ab. cat.

$\rightsquigarrow$ get add. $\hat{F}: Ch(\mathcal{A}) \to Ch(\mathcal{B})$, $A^\bullet \mapsto (\dots \to F(A^n) \xrightarrow{F(d^n)} F(A^{n+1}) \to \dots)$

$\rightsquigarrow$ get triang. $\hat{F}: K(\mathcal{A}) \to K(\mathcal{B})$

If F is exact, then $\hat{F}$ preserves q-isom.

$\rightsquigarrow$ get $F: D(\mathcal{A}) \to D(\mathcal{B})$

<u>But</u>: $\hat{F}$ does not preserve q-isom. in general

Let F be left exact., e.g. $Hom(-, A)$, $\Gamma(-)$, $- \otimes_R M$ ← right exact

<u>Def</u>:
A class $\mathcal{R} \subset ob(\mathcal{A})$ is <u>adapted</u> to F if
(i) $\mathcal{R}$ is stable under direct sums
(ii) F applied to acyclic complex in $\mathcal{R}$ is acyclic again
(iii) $\forall A \in \mathcal{A}\ \exists\ 0 \to A \to R$, for $R \in \mathcal{R}$.

<u>Prop</u>:
Let $\mathcal{R}$ be adapted to F and $S_\mathcal{R} :=$ q-isom. in $K^+(\mathcal{R})$.
Then $S_\mathcal{R}$ is a mult. system in $K^+(\mathcal{R})$ and the can.

$$S_\mathcal{R}^{-1} K^+(\mathcal{R}) \longrightarrow D^+(\mathcal{A})$$

is an equiv. of cat.

Then define

$$RF: D^+(\mathcal{A}) \xrightarrow{\sim} S_\mathcal{R}^{-1} K^+(\mathcal{R}) \xrightarrow{F \text{ comp.wise, by (ii)}} D^+(\mathcal{B})$$

triang. functor.

<u>Rmks</u>:
- Can/Should define RF via univ. property.
- $\mathcal{A}$ has enough $\underbrace{\text{inj. objs}}_{=: \mathcal{I}}$ $\Rightarrow$ $\mathcal{I}$ is adapted to any left exact functor

- Had ad hoc def. of R^iF, $i \geq 0$:

$A \in \mathcal{A} \rightsquigarrow 0 \to A \to I^0 \to I^1 \to \dots$ inj. res.

$R^iF(A) := H^i(F(I^0) \to \dots)$ ← deg 0

We see:

$R^iF(A) = H^i(RF(A))$

RF triang. $\rightsquigarrow$ long exact seq. for R^iF

Examples:

(1) X top. space, $D(X) = D(Sh_X)$

Fact: Sh_X has enough inj objs.

global sect. functor $\Gamma: Sh_X \to Ab$, left exact

$\rightsquigarrow R\Gamma: D^+(X) \to D^+(Ab)$, $H^i(R\Gamma(\mathcal{F})) = H^i(X, \mathcal{F})$

(2) External Hom

$Hom: \mathcal{A}^{op} \times \mathcal{A} \to Ab$, $(A, B) \rightsquigarrow Hom_{\mathcal{A}}(A, B)$

left exact in both arguments

Extend to bitriang. bifunctor

$Hom: K(\mathcal{A})^{op} \times K(\mathcal{A}) \to K(Ab)$, $Hom(A^\bullet, B^\bullet)^i = \prod_{j \in \mathbb{Z}} Hom_{\mathcal{A}}(A^j, B^{i+j})$

Thm:

If $\mathcal{A}$ has enough inj. (proj) objs, get bitriang.

$RHom: D^-(\mathcal{A})^{op} \times D^+(\mathcal{A}) \to D(Ab)$.

Moreover

$\underbrace{H^n(RHom(A^\bullet, B^\bullet))}_{=: Ext^n(A^\bullet, B^\bullet)} \cong Hom_{D(\mathcal{A})}(A^\bullet, B^\bullet[n])$