The Magnus embedding is a quasi-isometry

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Introduction

The Magnus embedding is the main tool for studying *free solvable groups*:

- The n^{th} *derived (commutator) subgroup* of a group G is

$$G^{(n)} = [G^{(n-1)}, G^{(n-1)}],$$

where $G^{(1)} = G' = [G, G] = \langle [g, g'] \mid g, g' \in G \rangle$.

- The *free solvable group* $S_{d,r}$ of degree d and rank r is given by

$$S_{d,r} = {}^{F_r}\big/_{F_r^{(d)}}.$$

- The Magnus embedding is a map $\phi : S_{d,r} \hookrightarrow \mathbb{Z}^r \wr S_{d-1,r}$.

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Quasi-isometries

- A *quasi-isometric embedding* f between two metric spaces (X, d_X) and (Y, d_Y) is an injective map $f : X \to Y$ such that there exist constants $C_1, \dots, C_4 > 0$ for which

$$C_1 d_X(x_1, x_2) - C_2 \leq d_Y(f(x_1), f(x_2)) \leq C_3 d_X(x_1, x_2) + C_4$$

  for any $x_1, x_2 \in X$.
- For us, X, Y are groups and d_X, d_Y are the corresponding word metrics.
- Wlog, $f : G \to H$ is a quasi-isometry if it preserves geodesic length "up to linearity".

Set-up and notation

- $F = \langle x_1, \ldots, x_r \rangle$
- $N \lhd F$
- $N' = [N, N] = \langle [x, y] \mid x, y \in N \rangle$
- $A = \langle a_1, \ldots, a_r \rangle$ – free abelian
- $B = {}^{F}/_{N}$

$$\begin{tikzcd} F \arrow[r, "\mu"] \arrow[d, "-"'] & {}^{F}/_{N'} \arrow[r, hook, "\phi"] \arrow[dl] & A \wr B \\ {}^{F}/_{N} & & \end{tikzcd}$$

- Show that ϕ is a quasi-isometry.

Geodesics in $A \wr B$

Wreath products

The *restricted wreath product* is the group:

$$A \wr B = \{bf \mid b \in B,\ f \in A^{(B)}\},$$

with multiplication defined by

$$bf \cdot cg = bc\, f^c g,$$

where

- $f^c(x) = f(xc^{-1})$ for $x \in B$.
- $A^{(B)}$ is the set of all functions from B to A of *finite support*
- Multiplication in $A^{(B)}$ is given by $f \cdot g(x) = f(x)g(x)$
- $1_{A^{(B)}}$ is the function $1 : B \to 1_A$.

Remark. B acts on $A^{(B)}$, so $A \wr B \simeq B \ltimes A^{(B)}$

A presentation for $A \wr B$

Let $A = \langle X \mid R_A \rangle$, $B = \langle Y \mid R_B \rangle$. Then

$$A \wr B = \left\langle X \cup Y \mid R_A,\ R_B,\ [a_1^{b_1}, a_2^{b_2}] \right\rangle,$$

where $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

- Define $f_{a,b}(x) = \begin{cases} a & \text{if } x = b \\ 1 & \text{otherwise} \end{cases}$
- Then $A \hookrightarrow A \wr B$ (via $a \mapsto f_{a,1}$)
- Any function $f \in A^{(B)}$ can be given as $\{(b_1, a_1), \ldots, (b_n, a_n)\}$
- Equivalently, $f = f_{a_1,b_1} \ldots f_{a_n,b_n} = f_{a_1,1}^{b_1} \ldots f_{a_n,1}^{b_n} \leftrightsquigarrow a_1^{b_1} \ldots a_n^{b_n}$

Geodesics in $A \wr B$

- Let w be a word in the generators of A and B. Rewrite it as

$$w = b\, A_1^{B_1} \dots A_k^{B_k},$$

$A_1, \dots, A_k \neq 1$ and $B_1, \dots, B_k$ are distinct.

Theorem (Parry)

$$\|w\|_{A \wr B} = \|b\|_B + \sum_{i=1}^{k} \|A_i\|_A + \mathcal{L}_{\mathrm{Cay}(B)}(B_1, \dots, B_k).$$

- $\mathcal{L}_{\mathrm{Cay}(B)}(B_1, \dots, B_k)$ is the length of a *minimum length cycle*: the shortest circuit in Cay(B) passing through $\{1, B_1, \dots, B_k\}$.

Two views of Fox derivatives

The Magnus embedding

- The Magnus embedding was originally defined as $\phi : F \to M$, where M is a matrix group with entries in a group ring.
- For a word w in generators $X = \{x_1, \ldots, x_r\}$,

$$\phi : {}^{F}\!/_{N'} \hookrightarrow A \wr {}^{F}\!/_{N}$$

is given by

$$\phi(w) = \overline{w} \cdot a_1^{\overline{\partial w/\partial x_1}} \ldots a_r^{\overline{\partial w/\partial x_r}}.$$

- Here, $\frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_r}$ are the *Fox derivatives* of w.

Fox derivatives

For any $x, y \in X$ the delta-function

$$\frac{\partial y}{\partial x} = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

extends linearly to a derivation $\frac{\partial}{\partial x} : \mathbb{Z}F \to \mathbb{Z}F$, called the *Fox partial derivative*.

Properties:

- **Product Rule.** $\dfrac{\partial uv}{\partial x} = \dfrac{\partial u}{\partial x} + u\dfrac{\partial v}{\partial x}$
- **Power Rule.** $\dfrac{\partial u^{-1}}{\partial x} = u^{-1}\dfrac{\partial u}{\partial x}$

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Example

Let $F = F(x_1, x_2)$ and $w = x_2^{-1}x_1x_2x_1^2x_2x_1^{-1}x_2^{-1}x_1^{-1}x_2$.

$$\frac{\partial w}{\partial x_1} = \frac{\partial x_2^{-1}}{\partial x_1} + x_2^{-1}\frac{\partial x_1x_2x_1^2x_2x_1^{-1}x_2^{-1}x_1^{-1}x_2}{\partial x_1}$$

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&= x_2^{-1} \left(\cancelto{1}{\frac{\partial x_1}{\partial x_1}} + x_1 \frac{\partial x_2 x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2}{\partial x_1} \right)
\end{aligned}
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&= x_2^{-1} + x_2^{-1}x_1\left(\frac{\partial x_2}{\partial x_1} + x_2\frac{\partial x_1^2x_2x_1^{-1}x_2^{-1}x_1^{-1}x_2}{\partial x_1}\right)
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&= x_2^{-1} + x_2^{-1}x_1\left(\cancel{\frac{\partial x_2}{\partial x_1}} + x_2\frac{\partial x_1^2x_2x_1^{-1}x_2^{-1}x_1^{-1}x_2}{\partial x_1}\right) \\
&= \dots \\
&= x_2^{-1} + x_2^{-1}x_1x_2 + x_2^{-1}x_1x_2x_1 - x_2^{-1}x_1x_2x_1^2x_2x_1^{-1} \\
&\quad -x_2^{-1}x_1x_2x_1^2x_2x_1^{-1}x_2^{-1}x_1^{-1}
\end{aligned}
$$

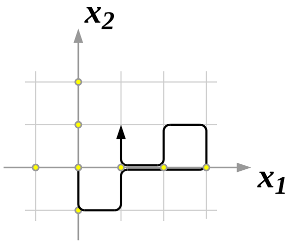
Flows in a Cayley graph

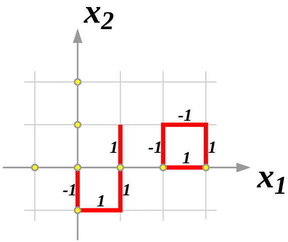
Consider the Cayley graph $\Gamma(G,X)$ as a digraph. Let $p = e_1 \dots e_n$ be a path in Γ and define a *flow* π_p as follows:

$$\pi_p(e) = \text{algebraic number of times that } p \text{ traverses } e.$$

Example of flows on $\Gamma(G, X)$

Example. Consider $G = F/F' \simeq \mathbb{Z} \times \mathbb{Z}$ with $X = \{x_1, x_2\}$. Find π_w for $w = x_2^{-1} x_1 x_2 x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2$.





Geometric interpretation of Fox derivatives

Edges in $\Gamma(F/N, X)$ have the form $e = (g, gx_i)$ for $g \in F/N$ and $i = 1, \ldots, r$.

Theorem (Miasnikov, Roman'kov, Ushakov, Vershik)

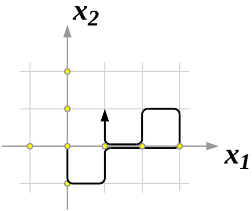
Let $w \in F$. Then

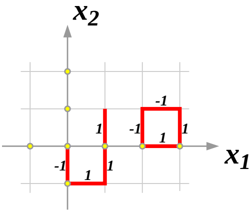
$$\overline{\frac{\partial w}{\partial x}} = \sum_{g \in F/N} \pi_w\big((g, gx)\big) g.$$

Example of Fox derivatives and flows

$$\frac{\partial w}{\partial x_1} = x_2^{-1} + x_2^{-1}x_1x_2 + x_2^{-1}x_1x_2x_1 - x_2^{-1}x_1x_2x_1^2x_2x_1^{-1}$$
$$-x_2^{-1}x_1x_2x_1^2x_2x_1^{-1}x_2^{-1}x_1^{-1}$$

$$\overline{\frac{\partial w}{\partial x_1}} = x_2^{-1} + \cancel{x_1} + x_1^2 - x_1^2x_2 - \cancel{x_1} = x_2^{-1} + x_1^2 - x_1^2x_2$$





Geodesics in F/N'

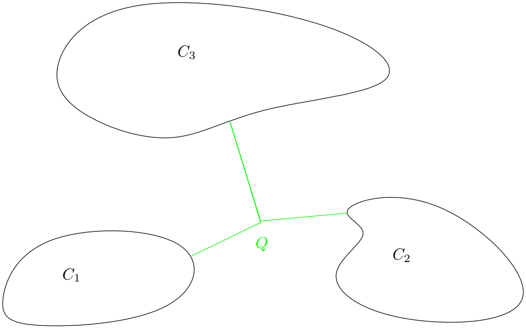
Finding geodesics

Goal. For a word $w \in F/N'$ given as a product of generators X, find a geodesic for w.

- Read w as a path p_w in Cay$({}^F/_N, X)$. (This is not a typo! N, not N'.)
- This path defines a flow, π_w.
- Consider the subgraph Γ of Cay$({}^F/_N, X)$ which consists of edges of non-zero flow.

Γ *and the minimal forest*

- $C_1, \ldots, C_l$ – connected components of Γ
- Q – minimal forest connecting $C_1, \ldots, C_l$
- $\Delta = Q \cup C_1 \ldots \cup C_l$



Note. There may be more than one choice for Q.

From Δ to Δ^* and back

Consider $\mathbb{Z} \times \mathbb{Z} = \langle x, y \rangle$.

$w = yxy^{-1}x^{-1}yxyx^2y^{-1}xyx^{-3}y^{-2}$

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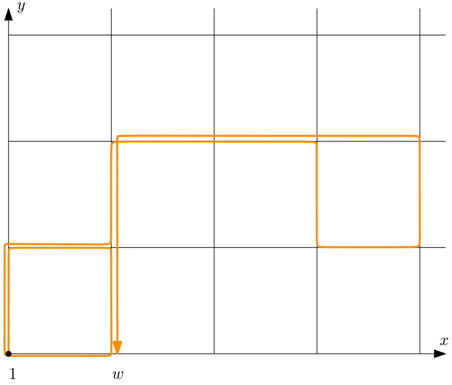

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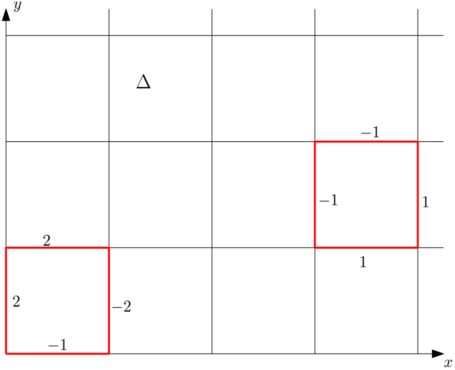

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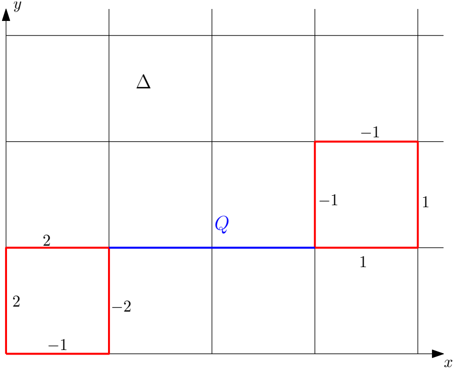

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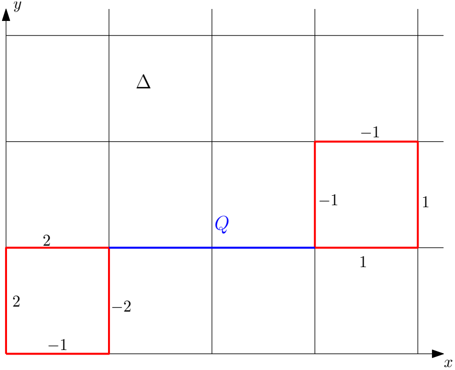
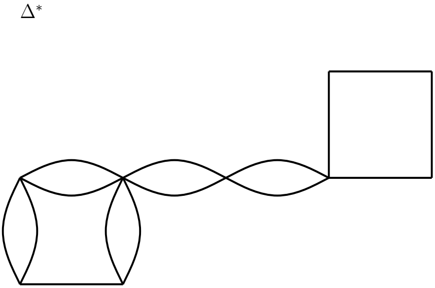




Geodesics in F/N'

An Euler tour on the vertices of Δ^* corresponds to a geodesic for w in ${}^{F}/_{N'}$.

Theorem. (Miasnikov, Roman'kov, Ushakov, Vershik)

$$\|w\|_{F/N'} = \sum_{e \in \mathrm{supp}(p_w)} |\pi_w(e)| + 2|E(Q)|.$$

The Magnus embedding is a quasi-isometry

Main Theorem

Theorem (V)

Let w be an element of ${}^{F}/_{N'}$ given as a product of generators $x_1, \ldots, x_r$. Then

$$\frac{1}{2(r+1)} \|w\|_{F/_{N'}} \leq \|\phi(w)\|_{A \wr B} \leq 3\|w\|_{F/_{N'}}.$$

Example

- $F = F(x, y), N = F'$
- ${}^{F}\!/_{N'} \simeq M_2$, free metabelian group
- $B = {}^{F}\!/_{N} \simeq \mathbb{Z} \times \mathbb{Z}$
- $A = \langle a_1, a_2 \rangle$ - free abelian group

- $w = yxy^{-1}x^{-1}yxyx^2y^{-1}xyx^{-3}y^{-2}$
- $\overline{{}^{\partial w}\!/_{\partial x}} = -1 + 2y + x^3y - x^3y^2$, $\overline{{}^{\partial w}\!/_{\partial y}} = 2 - 2x - x^3y + x^4y$
- $$\begin{aligned} \phi(w) &= \overline{w} \cdot a_1^{\overline{\partial w/\partial x}} a_2^{\overline{\partial w/\partial y}} = x \cdot a_1^{-1+2y+x^3y-x^3y^2} a_2^{2-2x-x^3y+x^4y} \\ &= x \cdot (a_1^{-1}a_2^2)(a_1^2)^y(a_2^{-2})^x(a_1a_2^{-1})^{x^3y}(a_1^{-1})^{x^3y^2}(a_2)^{x^4y} \end{aligned}$$

A crucial lemma

- $\phi(w) = \overline{w} \cdot A_1^{B_1} \ldots A_k^{B_k}$.
- $\sum_i \|A_i\|_A$ is the sum of absolute values of the coefficients in the Fox derivatives

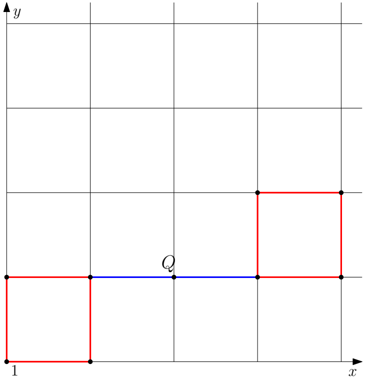
Lemma

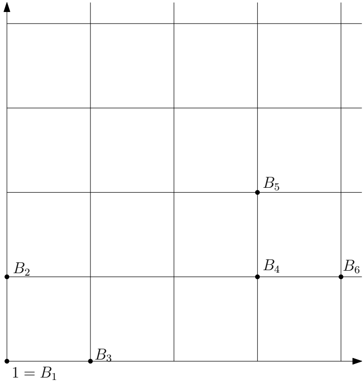
$$\sum_i \|A_i\|_A = \sum_{e \in \mathrm{supp}(p_w)} |\pi_w(e)|.$$

Proof by example

- $w = yxy^{-1}x^{-1}yxyx^2y^{-1}xyx^{-3}y^{-2}$
- $\overline{{\partial w}/{\partial x}} = -1 + 2y + x^3y - x^3y^2,$
- $\overline{{\partial w}/{\partial y}} = 2 - 2x - x^3y + x^4y$
- $\phi(w) = x \cdot \underbrace{(a_1^{-1}a_2^2)}_{A_1^{B_1}}\underbrace{(a_1^2)^y}_{A_2^{B_2}}\underbrace{(a_2^{-2})^x}_{A_3^{B_3}}\underbrace{(a_1a_2^{-1})^{x^3y}}_{A_4^{B_4}}\underbrace{(a_1^{-1})^{x^3y^2}}_{A_5^{B_5}}\underbrace{(a_2)^{x^4y}}_{A_6^{B_6}}$
- $\sum_{i=1}^{6} \|A_i\|_A = (1+2)+2+2+(1+1)+1+1 = \sum_{e \in \mathrm{supp}(p_w)} |\pi_w(e)|$

Comparing graphs – first look





- $g \in {}^{F}\!/_{N}$ such that $\left\{ \begin{array}{c} \pi_w(g, gx) \neq 0 \\ \text{or} \\ \pi_w(g, gy) \neq 0 \end{array} \right. \longleftrightarrow B_i$ for some i.

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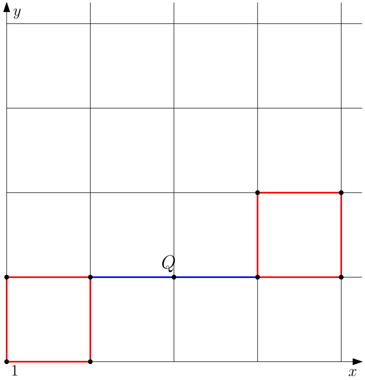

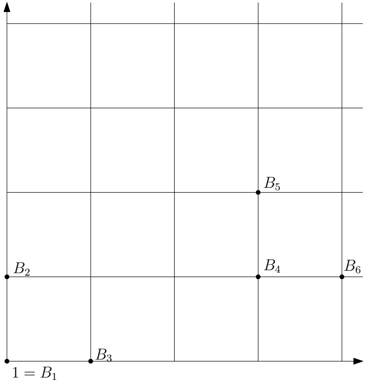

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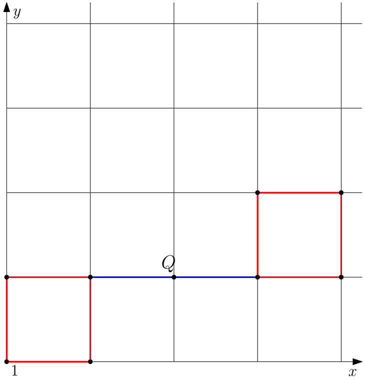

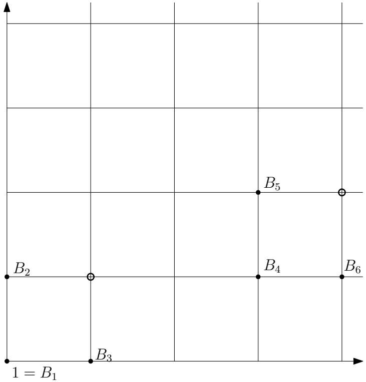

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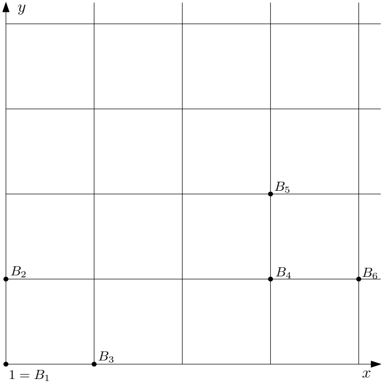
$\|\phi(w)\|_{A\wr B} \leq 3\|w\|_{F/N'}$

$$\|w\|_{F/N'} = \sum |\pi_w(e)| + 2|E(Q)| \qquad \|\phi(w)\|_{A\wr B} = \|\overline{w}\|_{F/N} + \sum \|A_i\| + |\mathcal{T}|$$

$$= \sum_{i=1}^{l} |E(C_i)| + 2|E(Q)|$$

- Any tour on $V(\Delta)$ is longer than a minimal tour $\mathcal{T}$ on $\{1, B_1, \ldots, B_k\}$, so
$$|\mathcal{T}| \leq \|w\|_{F/N'}$$
- $\sum \|A_i\| = \sum |\pi_w(e)|$, so
$$\sum \|A_i\| \leq \|w\|_{F/N'}$$
- $\|\overline{w}\|_{F/N} \leq \|w\|_{F/N'}$
- $\implies \|\phi(w)\|_{A\wr B} \leq 3\|w\|_{F/N'}$

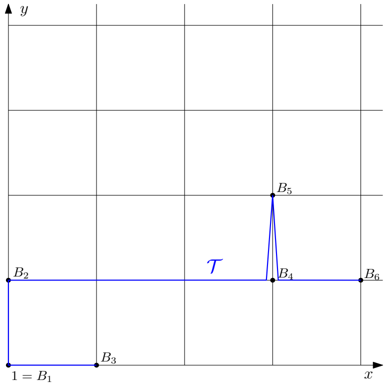
$\|w\|_{F/_{N'}} \leq (2r+1)\|\phi(w)\|_{A\wr B}$



$$\|w\|_{F/_{N'}} = \sum |\pi_w(e)| + 2|E(Q)| = \sum_{i=1}^{l} |E(C_i)| + 2|E(Q)|$$

$$\|\phi(w)\|_{A\wr B} = \|\overline{w}\|_{F/_N} + \sum \|A_i\| + |\mathcal{T}| = \sum \pi_w(e) + |\mathcal{T}|$$

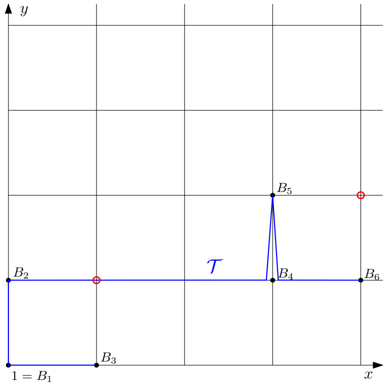
$$\|w\|_{F/N'} \leq (2r+1)\|\phi(w)\|_{A \wr B}$$



$$\|w\|_{F/N'} = \sum |\pi_w(e)| + 2|E(Q)| = \sum_{i=1}^{l} |E(C_i)| + 2|E(Q)|$$

$$\|\phi(w)\|_{A \wr B} = \|\overline{w}\|_{F/N} + \sum \|A_i\| + |\mathcal{T}| = \sum \pi_w(e) + |\mathcal{T}|$$

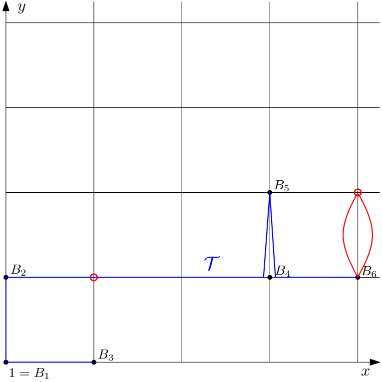
$$\|w\|_{F/N'} \leq (2r+1)\|\phi(w)\|_{A\wr B}$$



$$\|w\|_{F/N'} = \sum |\pi_w(e)| + 2|E(Q)| = \sum_{i=1}^{l} |E(C_i)| + 2|E(Q)|$$

$$\|\phi(w)\|_{A\wr B} = \|\overline{w}\|_{F/N} + \sum \|A_i\| + |\mathcal{T}| = \sum \pi_w(e) + |\mathcal{T}|$$

$$\|w\|_{F/N'} \leq (2r+1)\|\phi(w)\|_{A\wr B}$$



$$\|w\|_{F/N'} = \sum |\pi_w(e)| + 2|E(Q)| = \sum_{i=1}^{l} |E(C_i)| + 2|E(Q)|$$

$$\|\phi(w)\|_{A\wr B} = \|\overline{w}\|_{F/N} + \sum \|A_i\| + |\mathcal{T}| = \sum \pi_w(e) + |\mathcal{T}|$$