The Asphericity of Injective Labeled Oriented Trees

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Introduction

Joint work with Jens Harlander (Boise, Idaho, USA)

The Whitehead-Conjecture

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Labeled Oriented Trees

A LOG (labeled oriented graph) is a finite presentation (or the corresponding 2-complex) of the form:
$< x_1, \ldots, x_n \mid x_i x_j = x_j x_k, \ldots >$

Define an oriented graph:
Vertices $\longleftrightarrow$ Generators, Edges $\longleftrightarrow$ Relators
$< a, b, c, d, e \mid ac = cb, bd = dc, db = bc, da = ae >$

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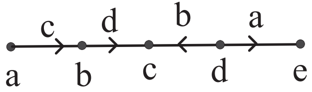

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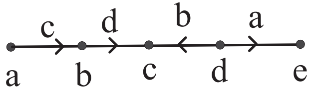

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Theorem (Howie 1983): Let L be a finite 2-complex and $e \subset L$ a 2-cell.

If $L \overset{3}{\nearrow\searrow} *$ $\Rightarrow$ $L - e \overset{3}{\nearrow\searrow} K$ and K is a LOT complex.

Andrews-Curtis Conjecture (AC): Let L be a finite, contractible 2-complex. Then $L \overset{3}{\nearrow\searrow} *$.

Corollary: (AC), LOTs are aspherical $\Rightarrow$ There is no finite counterexample $K \subset L$, L contractible, to (WH).
(The finite case)

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A nonaspherical LOT is a counterexample to (WH):

Any LOT is a subcomplex of an aspherical 2-complex (add $x_1 = 1$ as a relator. Can then be 3-deformed to a point).

Hence: The asphericity of LOTs is interesting for (WH)!

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Spherical diagrams

$f\colon C \to K^2$ is a *spherical diagram*, if C is a cell decomposition of the 2-sphere and open cells are mapped homeomorphically.

If K is non-aspherical then there exists a spherical diagram which realizes a nontrivial element of $\pi_2(K)$.

A spherical diagram $f\colon C \to K^2$ is *reducible*, if there is a pair of 2-cells in C with a common edge t, such that both 2-cells are mapped to K by folding over t.

A 2-complex K is said to be *diagrammatically reducible* (DR), if each spherical diagram over K is reducible.

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A LOT is called *injective* if each generator occurs at most once as an edge label (corresponds to alternating knots).

A LOT is called *compressed* if every relator contains 3 different generators.

A LOT is called *boundary-reducible* if there is a generator that occurs exactly once upon the set of relators. (A boundary vertex of a LOT which does not appear as edge label.)

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A result

Let P be a LOT. A *Sub-LOT* Q of P is a subtree of P such that it is a LOT itself (each edge label of Q is also a vertex label of Q).

Theorem 1 (Huck/Rosebrock 2001): If a compressed injective LOT P does not contain a boundary-reducible Sub-LOT then $K(P)$ (the corresponding 2-complex) is DR.

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Idea of Proof

Idea of Proof:

Let $K(P)$ be a 2-complex corresponding to a presentation P. The *Whitehead-Graph* $W(P)$ is the boundary of a regular neighborhood of the only vertex of $K(P)$.

Consists of a pair of vertices x_i^+ (beginning) and x_i^- (end) for each generator x_i.

The *left graph* $L \subset W(P)$ is the full subgraph on the vertices $x_1^+, \ldots, x_n^+$, the *right graph* $R \subset W(P)$ is the full subgraph on the vertices $x_1^-, \ldots, x_n^-$.

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Let $K(P)$ be a 2-complex corresponding to a presentation P. Let E be the set of edges of the Whitehead-Graph $W(P)$.

The weight test is satisfied for $K(P)$ if there is a *weight function* $g\colon E \to \mathbb{R}$, such that

1. the sum of the weights of every reduced cycle is ≥ 2 and
2. For every 2-cell $D \in K(P)$ whose boundary consists of d edges the sum of the weights of the corners of $W(P)$ that correspond to the corners of D is less than or equal to $d - 2$.

Theorem (Gersten) If the weight test is satisfied then $K(P)$ is DR.

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An *orientation* of a LOT P is a LOT Q that arises from P by changing the orientation of a subset of the edges of P.

Lemma 2: If the left graph and the right graph of a compressed injective LOT P are trees then any orientation of P is DR.

Idea of Proof: Changing the orientation does not change the isomorphism-type of the Whiteheadgraph of an injective LOT. If the left and the right graph are trees then the weight-test is satisfied which implies DR. The weight-test depends on the Whiteheadgraph and on the edges each 2-cell contributes to the Whiteheadgraph only. □

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For a compressed injective LOT P which does not contain a boundary-reducible Sub-LOT an orientation is found such that the left and the right graph are trees.

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The asphericity of injective LOTs

Theorem 3 (Harlander/Rosebrock 2012): Let P be a compressed injective LOT. Then $K(P)$ is DR.

In fact we show:

Theorem 4 Let P be a compressed LOT with maximal proper boundary-reducible sub-LOTs $T_1, \ldots, T_n$. Let P' be the LOT where each T_i is identified to a vertex t_i (in the underlying tree). Assume that each $K(T_i)$ is DR and that P' is injective. Then $K(P)$ is DR.

Theorem 3 follows by induction from Theorem 4 and Theorem 1.

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We follow the proof with an example:

Is injective and contains a reducible sub-LOT. In fact it does not satisfy the weight test (can be shown with software GRAPH).

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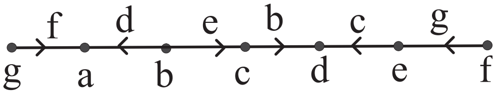

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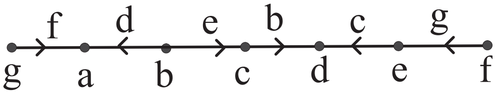

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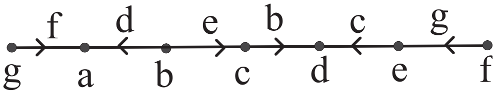

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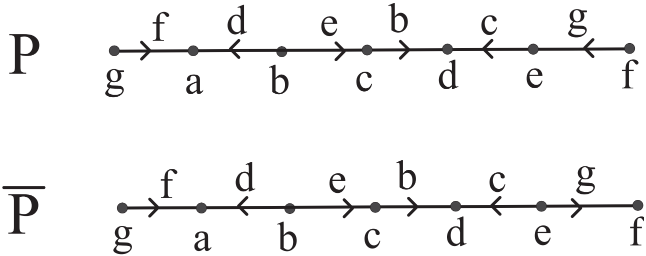

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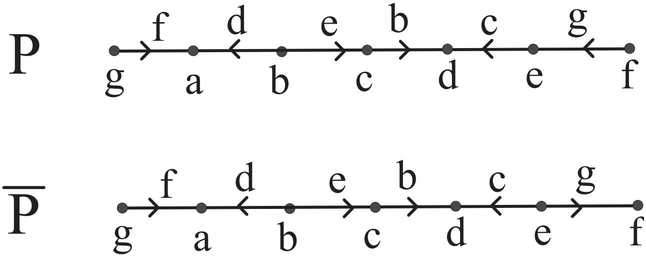

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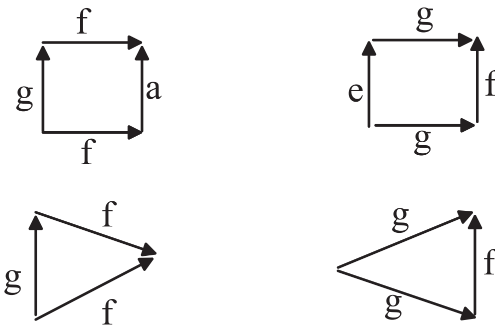

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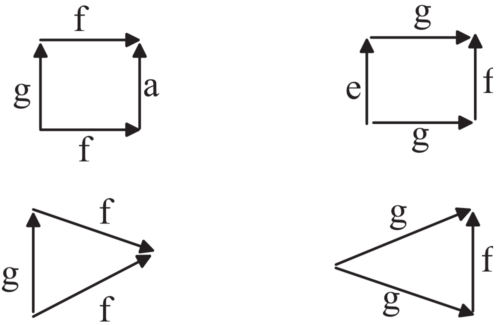
Given the LOT P with proper boundary-reducible sub-LOTs $T = \{T_1, \ldots, T_n\}$ we identify T to a single vertex in $K(\bar{P})$ to achieve the relative complex $K(\bar{P}/T)$.



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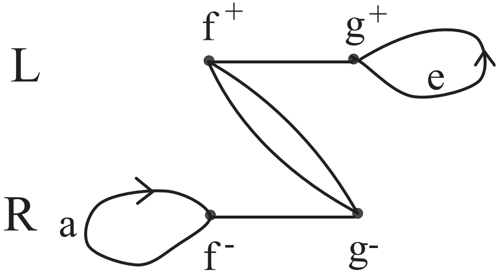

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Let $H = \pi_1(K(T))$. If $G_i = \pi_1(K(T_i))$, then $H = G_1 * \ldots * G_n$.

A cycle $c \in W(\bar{P}/T)$ is called *admissible* if the word $w(c)$ read from its corners is trivial in H.

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$K(P/T)$ *satisfies the relative weight test* if there is a real number $g(e)$, the *weight*, assigned to each corner (edge) $e \in W(P/T)$ such that

1. the sum of the weights of every reduced admissible cycle in $W(P/T)$ is ≥ 2, and
2. for every 2-cell $D \in K(P/T)$ whose boundary consists of d edges the sum of the weights of the corners of $W(P/T)$ that correspond to the corners of D is less than or equal to $d - 2$.

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Shown by Bogley/Pride (more general):

Theorem 5 Let P be a LOT and $T = \{T_1, \ldots, T_n\}$ a set of disjoint sub-LOTs of $T(P)$. If $K(P/T)$ satisfies the relative weight test and all the $K(T_i)$ are DR then $K(P)$ is DR.

(Idea of Proof: If $K(P)$ is not DR then there is a reduced spherical diagram $f\colon C \to K(P)$. This cannot map to $K(T)$ only because $K(T)$ is DR. So it can be transformed into a spherical diagram $f_{\#}\colon C \to K(P/T)$ but this contradicts the weight test.)

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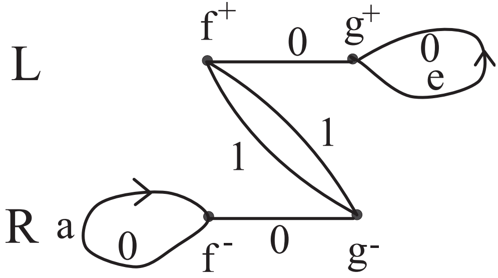

The asphericity of injective LOTs

We assign weight 0 to the edges of L and R.

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So condition 2 of the relative weight test is satisfied also and the relative weight test is satisfied for $K(\bar{P}/T)$.

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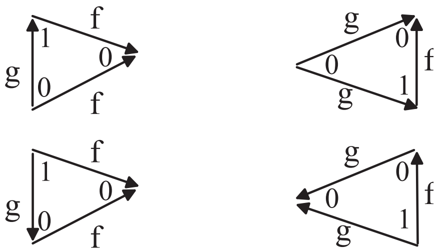

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Thank you for your attention

- Guenther Huck and Stephan Rosebrock.
  Aspherical Labelled Oriented Trees and Knots,
  Proceedings of the Edinburgh Math. Soc. 44 (2001).
- Jens Harlander and Stephan Rosebrock.
  Generalized knot complements and some aspherical ribbon disc complements,
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