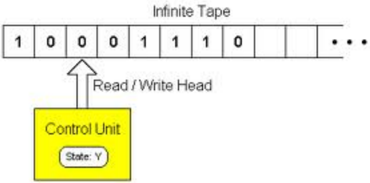
# Residually finite finitely presented solvable groups

Olga Kharlampovich (Joint results with A. Miasnikov and M.Sapir)

August 2012, Dusseldorf

# Two 100 anniversaries

The word problem in groups becomes 100 years old this year (Max Dehn introduced it in 1912). This year the mathematical world also celebrates the 100'th anniversary of Alan Turing.




# Abstract

We construct the first examples of finitely presented residually finite groups with arbitrarily complicated word problem and depth function. The groups are solvable of class 3. We also prove that the universal theory of finite solvable of class 3 groups is undecidable.

# McKinsey, Malcev algorithm

# McKinsey, Malcev algorithm

Let $G = \langle X; R \rangle$ be a residually finite finitely presented algebraic structure of finite type (signature) $T$ (say, groups, semigroups, rings, etc.) Let $F(X)$ be the free algebraic structure of type $T$ freely generated by $X$. Then we define the "yes" and "no" parts of the word problem in $G$ as follows:

$$\mathrm{WP}_{\mathrm{yes}} = \{(w, w') \in F^2(X) \mid w =_G w'\} \text{ and}$$
$$\mathrm{WP}_{\mathrm{no}} = \{(w, w') \in F^2(X) \mid w \neq_G w'\}.$$

To solve the word problem in $G$ one runs in parallel two separate algorithms $\mathcal{A}_{\mathrm{yes}}$ and $\mathcal{A}_{\mathrm{no}}$, such that starting with a given pair of elements $w, w' \in F(X)$ $\mathcal{A}_{\mathrm{yes}}$ stops if and only if $(w, w') \in \mathrm{WP}_{\mathrm{yes}}$ and $\mathcal{A}_{\mathrm{no}}$ stops if and only if $(w, w') \in \mathrm{WP}_{\mathrm{no}}$.
**The algorithm** $\mathcal{A}_{\mathrm{yes}}$ enumerates one by one all consequences of the defining relations $R$ and waits until $w = w'$ appears in the list.
**The algorithm** $\mathcal{A}_{\mathrm{no}}$ enumerates all homomorphisms $\phi_1, \phi_2, \ldots,$ of $G$ into finite algebraic structures of type $T$ and waits until $\phi_i(w) \neq \phi_i(w')$.

# The most common residually finite groups

The most "common" residually finite groups are linear groups, say, over fields (Malcev). In that case the word problem can be solved in deterministic polynomial time (Lipton, Zalstein, Waak). The "no" part can be solved by considering factor groups corresponding to ideals of finite index of some polynomial rings, hence also can be shown to be solvable in deterministic polynomial time.
The same can be said about most finitely presented groups (where "most" means "overwhelming probability" in one of several probabilistic models): most finitely presented groups have small cancellation, (Olshanskii), small cancellation groups are virtually RAAGs (Ollivier, Wise, Agol) which are linear (even over $\mathbb{Z}$).

**Theorem.** (Kh., Myasn., Sapir) Let $f(n)$ be a recursive function. Then there exists a residually finite finitely presented solvable group $G$ such that for any finite presentation $\langle X; R \rangle$ of $G$ the time complexity of both "yes" and "no" parts of the word problem are at least as high as $f(n)$.

# Quantification of the “yes” part: the word problem

Madlener and Otto: in the case of a group or semigroup $G$ the complexity of the **non-deterministic** algorithm $\mathcal{A}_{\text{yes}}$ can be characterized by the Dehn function of $G$.

**Definition** (Madlener-Otto, Gersten, Gromov) Let $G = \langle X|R\rangle$ be a f.p. group, $w$ be a word in $X$, $w = 1$ in $G$. The *area* of $w$ is the minimal number of cells in a van Kampen diagram with boundary label $w$. By other words, this is the minimal number $n$ such that

$$w = \Pi_{i=1}^{n} g_i r_{j_i} g_i^{-1},$$

where $r_{j_i} \in R$, in the free group with basis $X$.

**Definition (Dehn function)** For any $n \geq 1$ let $d(n)$ be the largest area of a word $w$ of length at most $n$.

# Quantification of the “yes” part: the word problem

Gersten: the question about possible Dehn function of a residually finite group.
Nilpotent groups are examples of residually finite groups with arbitrary high polynomial Dehn function. The Baumslag-Solitar groups $\langle x, y \mid x^y = x^k \rangle$, $k \geq 2$, are residually finite (even linear) groups with exponential Dehn function. It was also known that super-exponential behavior can occur, although it does not seem to happen in “natural” classes of groups. Thus if $G$ is a finitely presented group with an unsolvable word problem, then the Dehn function of $G$ cannot be bounded above by any recursive function.

**Theorem** (Sapir, Birget, Rips). Let $L$ be a language accepted by a Turing machine $M$ with a superadditive time function $T(n) > n^4$. Then there exists a f.p. group $G$ with the Dehn function equivalent to $T(n)$.

In Gersten-Baumslag's group $< x, y | x^{(x^y)} = x^2 >$ Dehn function is not bounded by a tower of exponents but the W.P. is polynomial (Myasnikov, Ushakov).

## Our result

**Theorem** (KMS) For every recursive function $f$, there is a residually finite finitely presented solvable of class 3 group $G$ with Dehn function greater than $f$. In addition, one can assume that the word problem in $G$ is at least as hard as the membership problem in a given recursive set of natural numbers $Z$ or as easy as polynomial time.

**Corollary.** For every recursive function $f$, there is a residually finite finitely presented solvable of class 3 group $G$ with Dehn function greater than $f$ and the word problem decidable in polynomial time.

# Quantification of the “no” part: the depth function

The function quantifying the algorithm $\mathcal{A}_{\mathrm{no}}$ is the *depth function* introduced by Bou-Rabee (2010). Recall that if $G = \langle X \rangle$ is a finitely generated group or semigroup, the depth function $\rho_G(n)$ is the smallest function such that every two words $w \neq_G w'$ of length at most $n$ are separated by a homomorphism to a group (semigroup) $H$ with $|H| \leq \rho_G(n)$. That function does not depend on the choice of finite generating set $X$ (up to the natural equivalence).

# Quantification of the "no" part: the depth function

For every finitely generated linear group or semigroup $G$, $\rho_G$ is at most polynomial. Since finitely generated metabelian groups are subgroups of direct products of linear groups (Wehrfritz, 80) the depth function of every finitely generated metabelian group is at most polynomial.

By the recent result of Agol (2012) based on the results of Wise (2011), every small cancelation group is a subgroup of a Right Angled Artin group, hence linear and has polynomial depth function.

For the free group $F_2$, $\rho_{F_2}(n)$ is at most $n^{\frac{2}{3}}$ by a result of Kassabov and Matucci (2012).

# Quantification of the "no" part: the depth function

There are some finitely presented groups for which the depth function is unknown and very interesting. For example the ascending HNN extensions of free groups are known to be residually finite and even virtually residually nilpotent (Borisov, Sapir, 2005) but the only upper bound one can deduce from the proof is exponential. Although many of these groups have small cancelation presentations and so covered by the results of Agol, there are some groups of this kind for which the depth function is not known.

One of these groups is $\langle x, y, t \mid txt^{-1} = xy, tyt^{-1} = yx \rangle$. It is hyperbolic. If the depth function of that group is not polynomial, that group would not be linear, disproving a conjecture by Wise (he conjectured that all hyperbolic ascending HNN extensions of free groups are linear and, moreover, subgroups of Right Angled Artin groups).

# Quantification of the “no” part: the depth function

For finitely generated infinitely presented groups (even amenable ones) the situation is much more clear now. Using the method of Kassabov and Nikolov (2009) and the result of Nikolov and Segal (2003) one can construct a finitely generated residually finite group with arbitrary large recursive depth function.

**Theorem** (KMS) For every recursive function $f$, there is a residually finite finitely presented solvable of class 3 group $G$ with depth function greater than $f$. In addition, one can assume that the word problem in $G$ is at least as hard as the membership problem in a given recursive set of natural numbers $Z$ or as easy as polynomial time.

**Corollary.** For every recursive function $f$, there is a residually finite finitely presented solvable of class 3 group $G$ with depth function greater than $f$ and the word problem decidable in polynomial time.

# Slogan

All bad examples that existed for f.p. groups we can construct for f.p. residually finite 3-step solvable groups.

# Subgroup distortion of pro-finitely closed subgroups of finitely presented groups

Let $G$ be a group generated by a finite set $X$, $H \leq G$ be a subgroup generated by a finite set $Y$. Recall that the distortion function $f_{H,G}(n)$ is defined as the minimal number $f$ such that every element of $H$ represented as a word $w$ of length $\leq n$ in the alphabet $X$ can be represented as a word of length $\leq f$ in the alphabet $Y$.

The distortion function $f_{G,H}$ is recursive if and only if the membership problem in $H$ is decidable.

## Subgroup distortion of pro-finitely closed subgroups of finitely presented groups

As usual we say that $H$ is *closed in the pro-finite topology* of $G$ if $H$ is the intersection of subgroups of $G$ of finite index.
If $G$ is finitely presented and $H$ is closed in the pro-finite topology of $G$, then there exists a McKinsey-type algorithm $A(G, H)$ solving the membership problem for $H$ (and thus the $f_{G,H}$ is recursive). For every word $w$ in the alphabet $X$, the "yes" part $A\mathrm{yes}(G, H)$ of the algorithm lists all words in $Y$, rewrites them as words in $X$, and then applies relations of $G$ to check whether one of these words is equal to $w$. The "no" part $A_{\mathrm{no}}(G, H)$ of the algorithm lists all homomorphisms $\phi$ of $G$ into finite groups and checks whether $\phi(w) \notin \phi(H)$. One can ask what is the complexity of the "yes" and "no" parts of that algorithm, in particular, and of the membership problem for $H$ in general.

One can also quantify the complexity of the two parts $A_{\rm yes}(G, H)$ and $A_{\rm no}(G, H)$. The "yes" part is quantified by the distortion function $f_{G,H}(n)$ and the "no" part is quantified by the *relative depth function* $\rho_{G,H}(n)$ which is defined as the minimal number $r$ such that for every word $w$ of length $\leq n$ in $X$ which does not represent an element of $H$ there exists a homomorphism $\phi$ from $G$ to a finite group of order $\leq r$ such that $\phi(w) \notin \phi(H)$.
There were no examples of finitely generated subgroups of finitely presented groups that are closed in the pro-finite topology but have "arbitrary bad" distortion or "arbitrary bad" relative depth function.

The well known Mihailova's construction shows that finitely generated subgroups of the residually finite group $F_2 \times F_2$ (here $F_2$ is a free group of rank 2) could be as distorted as one pleases. In fact the set of possible distortion functions of subgroups of $F_2 \times F_2$ coincides, up to a natural equivalence, with the set of Dehn functions of finitely presented groups (Olshanskii, Sapir, 2001).

The *equalizers* of pairs of homomorphisms $\phi\colon F_k \to G, \psi\colon F_n \to G$ (where $F_k, F_n$ are subgroups of $F_2$) are the subgroups of $F_k \times F_n$ of the form $\{(x, y) \in F_k \times F_n \mid \phi(x) = \psi(y)\}$. The equalizer subgroup is finitely generated if and only if $G$ is finitely presented. It is easy to prove that if $G$ is residually finite, then the equalizer is closed in the pro-finite topology of $F_2 \times F_2$.

**Theorem** (KMS) For every recursive function $f(n)$ there exists a finitely generated subgroup $H \leq F_2 \times F_2$ that is closed in the pro-finite topology of $F_2 \times F_2$ and whose distortion function $f_{F_2 \times F_2, H}$, the relative depth function, and the time complexities of both "yes" and "no" parts of the membership problem are at least $f(n)$.

There is an analogous (though a bit weaker) result, for subgroups of a direct product $S_3(X) \times S_3(X)$, where $S_3(X)$ is a free solvable group of class 3 with free generating set $X$.

**Theorem** (KMS) For any recursive function $f(n)$ there is a finite set $X$ and a finitely generated subgroup $H \in S_3(X) \times S_3(X)$ such that $E$ is closed in the pro-finite topology on $S_3(X) \times S_3(X)$ and whose distortion function, the relative depth function, and the time complexities of both "yes" and "no" parts of the membership problem are at least $f(n)$.

# Methods

A possible idea to construct complicated residually finite finitely presented groups would be to take a complicated Turing machine with decidable halting problem and show that the corresponding group is residually finite. Unfortunately even for simple Turing machines the corresponding groups are not residually finite. Thus we have to modify the machine. We use the fact that every Turing machine with decidable halting problem is equivalent to a *universally halting* and even *sym-universally halting* Turing machine.

A machine $M$ in general has an alphabet and a set of words in that alphabet called *configurations*. It also has a finite set of commands. Each command is a partial injective transformation of the set of configurations. A computation is a sequence

$$w_1 \xrightarrow{\theta_1} w_2 \xrightarrow{\theta_2} \dots \xrightarrow{\theta_l} w_{l+1}.$$

where $w_j$ are configurations, $\theta_1, ..., \theta_n$ are commands and $\theta_i(w_i) = w_{i+1}$ for every $i = 1, ..., n$. A machine is called *deterministic* if the domains of its commands are disjoint. A machine usually has a distinguished *stop* configuration, and a set $I = I(M)$ of *input* configurations. A configuration is called *accepted* by $M$ if there exists a computation connecting that configuration with the stop configuration.

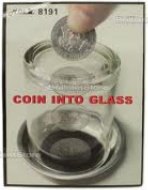
The machine $\mathrm{Sym}(M)$ is defined in the natural way (add the inverses of all commands of $M$).

A (not necessarily deterministic) machine $M$ is called *universally halting* if for every configuration $w$ of $M$ there exist only finitely many computations of $M$ starting with $w$ without repeated configurations.

We call a deterministic machine $M$ sym-universally halting if $\mathrm{Sym}(M)$ halts if it starts with any non-accepted configuration.

# Minsky machines

We are using (a modified version of) my construction to construct a finitely presented 3-solvable group with unsolvable word problem. This is an interpretation of a Minsky machine $M$ in finitely presented solvable groups $G(M)$ of class 3. The main feature of the group $G(M)$ is that the words corresponding to the configurations of $M$ and some of their subwords form a basis of the second derived subgroup $G(M)''$ of $G(M)$ which is an Abelian group of prime exponent (i.e. a vector space over $\mathbb{Z}/p\mathbb{Z}$). The factor-group $G(M)/G(M)''$ is metabelian, hence residually finite (and with easy word problem) by. Thus the "extra elements" of $G(M)$ are easy to deal with.




The hardware of a $K$-glass Minsky machine, $K \geq 2$, consists of $K$ glasses containing coins. We assume that these glasses are of infinite height. The machine can add a coin to a glass, and remove a coin from a glass (provided the glass is not empty). The commands of a Minsky machine are numbered. So a configuration of a $K$-glass Minsky machine is a $K+1$-tuple $(i; \epsilon_1, \ldots, \epsilon_K)$ where $i$ is the number of command that is to be executed, $\epsilon_j$ is the number of coins in the glass $\#j$.

More precisely, a *command* has one of the following forms:

- Put a coin in each of the glasses $\#\#n_1, ..., n_l$ and go to command $\#\ j$. We shall encode this command as

$$i; \to \mathrm{Add}(n_1, ..., n_l); j$$

  where $i$ is the number of the command;

- If the glasses $\#\#n_1, ..., n_l$ are not empty then take a coin from each of these glasses and go to instruction $\#\ j$. This command is encoded as

$$i; \epsilon_{n_1} > 0, ..., \epsilon_{n_l} > 0 \to Sub(n_1, ..., n_l); j;$$

- If glasses $\#\#n_1, ..., n_l$ are empty, then go to instruction $\#\ j$. This command is encoded as

$$i; \epsilon_{n_1} = 0, ..., \epsilon_{n_l} = 0 \to j;$$

- Stop. This command is encoded as $i; \to 0$;

**Remark** This defines deterministic Minsky machines. We will also need non-deterministic Minsky machines. Those will have two or more commands with the same number.

## Theorem

*Let $X$ be a r.e. set of numbers. Then the following holds:*
*(a) There exists a 3-glass deterministic Minsky machine $MM_3$ which when started on $(1; m, 0, \ldots, 0)$ stops in the configuration $(q_0, 0, 0, \ldots, 0)$ provided $m \in X$, and works forever otherwise.*
*(b) We can also assume that every computation of $MM_2$ or $MM_3$ starting with a configuration $c$ empties each glass after at most $O(|c|)$ steps.*
*(c) If $X$ is recursive, then the machine $MM_3$ above can be chosen to be sym-universally halting.*
*(d)If $M$ is a deterministic Turing machine recognizing $X$, then we can assume that $MM_2$ (resp. $MM_3$) polynomially reduces to $M$ where the numbers written on the tapes of $M$ are measured as represented in unary (that is the size of a number $n$ is set as $n$ and not $\log_2 n$).*

# The paper is in the Arxiv