On hyperbolicity of the free splitting and free factor complexes

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Based on joint work with Kasra Rafi

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Dr. Gillian Taylor: "Don't tell me, you're from outer space."
Captain Kirk: "No, I'm from Iowa. I only work in outer space."

The 1986 movie Star Trek IV: The Voyage Home

"Outer space is no place for a person of breeding."
Lady Violet Bonham Carter

"Interestingly, according to modern astronomers, space is finite. This is a very comforting thought - particularly for people who cannot remember where they left things."
Woody Allen

"Space is almost infinite. As a matter of fact, we think it is infinite."
Dan Quale

Plan

1. Curve complex for surfaces
2. Free splitting and free factor complexes for F_N
3. Statement of the main result
4. Bowditch criterion of hyperbolicity and its implications
5. Free bases graph
6. Sketch of the proof of the main result
7. Open problems (time permitting)

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Curve complex for surfaces.

Let S be a closed surface of negative Euler char. The *curve complex* $\mathcal{C}(S)$, introduced by Harvey in 1970s, has the vertex set consisting of free homotopy classes $[\alpha]$ of essential simple closed curves on S.

Two distinct vertices $[\alpha], [\beta]$ are joined by an edge if there exist disjoint representatives α, β of $[\alpha], [\beta]$. Higher-dimensional simplices are defined similarly.

The mapping class group $Mod(S)$ acts on $\mathcal{C}(S)$ by simplicial automorphisms.

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Curve complex for surfaces

Facts:
(1) $\mathcal{C}(S)$ is connected and $\dim \mathcal{C}(S) < \infty$
(2) $\mathcal{C}(S)$ is locally infinite
(3) $\mathcal{C}(S)$ has infinite diameter
(4) [Masur-Minsky, late 1990s]) $\mathcal{C}(S)$ is Gromov-hyperbolic.

The curve complex $\mathcal{C}(S)$ has many applications in the study of mapping class groups and of Teichmuller space, of Kleinian groups and of 3-manifolds.

Question: What about a free group F_N? Any "nice" complexes with natural $Out(F_N)$-action?

Several analogs of $\mathcal{C}(S)$ for F_N were suggested in recent years.

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Free splitting and free factor complexes

Defn. The *free splitting complex* FS_N has as its vertex set the set of "elementary free splittings" $F_N = \pi_1(\mathbb{A})$ where $\mathbb{A}$ is a (minimal nontrivial) graph of groups with a single edge (possibly a loop-edge) and the trivial edge group.Two such splittings are considered equal if their Bass-Serre trees are F_N-equivariantly isomorphic.

E.g. $F_N = A * B$ and $F_N = gAg^{-1} * gBg^{-1}$ are equal in FS_N.

Adjacency in FS_N corresponds to two splittings $F_N = \pi_1(\mathbb{A}_1)$ and $F_N = \pi_1(\mathbb{A}_2)$ admitting a *common refinement*, i.e. a splitting $F_N = \pi_1(\mathbb{B})$ where $\mathbb{B}$ has TWO edges e_1, e_2, both with trivial edge groups, and where for $i = 1, 2$ collapsing the edge e_i produces the splitting $F_N = \pi_1(\mathbb{A}_i)$.

E.g. if $F_N = A * B * C$ (with $A, B, C \neq \{1\}$) then the splittings $F_N = A * (B * C)$ and $F_N = (A * B) * C$ are adjacent vertices in FS_N.

Higher-dimensional simplices are defined similarly.

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Free splitting and free factor complexes

Facts. Let $N \geq 3$. Then:
(1) Both FS_N and FF_N are connected, finite-dimensional and admit natural co-compact $Out(F_N)$-actions.

(2) Both FS_N and FF_N are locally infinite.

(3) Both FS_N and FF_N have infinite diameter. (Kapovich-Lustig '09, Behrstock-Bestvina-Clay '10)

(4) If $\phi \in Out(F_N)$ is fully irreducible (iwip) then ϕ acts on FS_N and FF_N with positive asymptotic translation length (Bestvina-Feighn '10)

(5) There is a canonical $Out(F_N)$-equivariant coarsely Lipschitz and coarsely surjective "multi-function" $\tau : FS_N^{(0)} \to FF_N^{(0)}$ where $\tau(\mathbb{A})$ is the set of conjugacy classes of vertex groups of $\mathbb{A}$. The image $\tau(\mathbb{A})$ of a vertex of FS_N has diameter ≤ 2 in FF_N.

E.g. $\tau(F_N = A * B) = \{[A], [B]\}$.

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E.g. $\tau(F_N = A * B) = \{[A], [B]\}$.

Free splitting and free factor complexes

Facts. Let $N \geq 3$. Then:
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Theorem 1. [Bestvina-Feighn, July 2011, arXiv:1107.3308]
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Theorem 2. [Handel-Mosher, November 2011, arXiv:1111.1994]
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Thus we obtain:
Theorem 3. Let $N \geq 3$. Then:
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Bowditch's criterion of hyperbolicity and its consequences

Defn.[Thin structure] Let X be a connected graph with simplicial metric d_X. Let $\mathcal{G} = \{g_{x,y} | x, y \in V(X)\}$ be a family of edge-paths in X such that for any vertices x, y of X $\beta_{x,y}$ is a path from x to y in X.
Let $\Phi : V(X) \times V(X) \times V(X) \to V(X)$ be a function such that for any $a, b, c \in V(X)$,

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Then, we say that the pair $(\mathcal{G}, \Phi)$ is a (B_1, B_2)–*thin triangles structure* on X.

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The following statement is a direct corollary of a more general hyperbolicity criterion due to Bowditch (2006)

Proposition. *Let X be a connected graph. For every $B_1 > 0$ and $B_2 > 0$, there exist $\delta > 0$ and $H > 0$ so that if $(\mathcal{G}, \Phi)$ is a (B_1, B_2)–thin triangles structure on X then X is δ–hyperbolic.*
Moreover, every path $\beta_{x,y}$ in $\mathcal{G}$ is H–Hausdorff-close to any geodesic segment $[x, y]$.

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Bowditch's criterion of hyperbolicity and its consequences

From here we derive the following useful corollary:

Corollary A *For every $\delta_0 \ge 0$, $L \ge 0$, $M \ge 0$ there exist $\delta_1 \ge 0$ and $H \ge 0$ so that the following holds.*
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Let $f : X \to Y$ be an L–Lipschitz graph map. Suppose that:

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Free bases graph

We introduce the following useful object that is q.i. to FF_N:

Defn The *free bases graph* FB_N has as its vertex set the set of equivalence classes $[\mathcal{A}]$ of free bases $\mathcal{A}$ of F_N.
Two free bases $\mathcal{A}$ and $\mathcal{B}$ are equivalent if the Cayley graphs $Cay(F_N, \mathcal{A})$ and $Cay(F_N, \mathcal{B})$ are F_N-equivariantly isometric.
(E.g $\mathcal{A} \sim g\mathcal{A}g^{-1}$. Also, permuting elements of $\mathcal{A}$ and possibly inverting some of them preserves the equivalence class $[\mathcal{A}]$.)

Two distinct vertices $[\mathcal{A}]$ and $[\mathcal{B}]$ are adjacent in FB_N if there exist representatives $\mathcal{A}$ of $[\mathcal{A}]$ and $\mathcal{B}$ of $[\mathcal{B}]$ such that $\mathcal{A} \cap \mathcal{B} \neq \emptyset$.

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Then q *is a quasi-isometry between* FB_N *and* FF_N.

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Recall that FS'_N is Gromov-hyperbolic by Handel-Mosher.

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Instead of a geodesic $[x, y]$ in FS'_N can use a quasi-geodesic from x to y.

Handel-Mosher, given any vertices $x, y \in FS_N$, construct a "folding line" $g_{x,y}$ from x to y in FS'_N and show that $g_{x,y}$ is a (reparameterized) uniform quasigeodesic in FS'_N.

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Form a labelled graph Γ_0 which is a wedge of N loop-edges at a vertex v_0 with the i-th loop-edge being labelled by the freely reduced word w_i over $\mathcal{A}$ such that $w_i = b_i$ in F_N.Thus the 1-st loop-edge is labelled by a_1.

By conjugating $\mathcal{A}$ by a_1^t if necessary may achieve the following important technical condition, needed by the Handel-Mosher construction:

among the $2N$ oriented edges outgoing from v_0 in Γ_0, there exist some three edges with their labels beginning with three distinct letters from $\mathcal{A}^{\pm 1}$.

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Now construct a sequence of labelled graphs $\Gamma_0, \Gamma_1, \Gamma_2, \dots$ where each Γ_{i+1} is obtained from Γ_i by a "maximal fold":

There is a vertex v in Γ_i and two outgoing edges e_1, e_2 from v with labels w_1, w_2 such that the freely words $w_1, w_2 \in F(\mathcal{A})$ have the same first letter. The graph Γ_{i+1} is obtained from Γ_i by "folding" together into a single edge the initial segments of e_1, e_2 corresponding to the maximal common initial segment of the word w_1, w_2.

Since $\mathcal{B}$ and $\mathcal{A}$ are free bases of F_N, the sequence is guaranteed to terminate in a finite number of steps with $\Gamma_m = R_{\mathcal{A}}$, the graph with a single vertex and N loop-edges labelled $a_1, \dots, a_N$.

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Handel-Mosher's general results imply: the sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ determines a uniform quasigeodesic $g_{x,y}$ from $x = [\mathcal{B}]$ to $y = [\mathcal{A}]$ in FS'_N.

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Therefore FB_N is Gromov-Hyperbolic by Corollary A'. Hence FF_N is also Gromov-hyperbolic since FF_N is q.i. to FB_N by Prop 1.

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Open problems

Problem 1. Let $\mathcal{A}, \mathcal{B}$ be free bases of F_N. Again consider $[\mathcal{A}]$ and $[\mathcal{B}]$ as vertices of FS'_N.

Let $n = d_{FS'_N}([\mathcal{A}], [\mathcal{B}])$.

Let U be the set of all vertices of FS'_N that occur along all folding paths $\Gamma_0, \dots, \Gamma_m$ from $[\mathcal{B}]$ to $[\mathcal{A}]$ in FS'_N as in the proof of Thm 3.

Is it true that

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Recall that $\phi \in Out(F_N)$ is *fully irreducible* or *iwip* if there is no power ϕ^t ($t \neq 0$) such that ϕ^t fixes the conjugacy class of a proper free factor of F_N.

Fact: Let $\phi \in Out(F_N)$. Then exactly one of the following occurs:

- ϕ is an iwip and it acts as a hyperbolic isometry on FF_N (has a quasi-axis and exactly 2 fixed points at infinity)
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Another model: FS_N^* has $V(FS_N^*) = V(FS_N)$.
Two distinct vertices $\mathbb{A}, \mathbb{B}$ of FS_N^* are adjacent if there exists $w \in F_N, w \neq 1$ such that

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- ϕ is an iwip and it acts as a hyperbolic isometry on FF_N (has a quasi-axis and exactly 2 fixed points at infinity)
- ϕ is not an iwip and some nonzero power ϕ^t of ϕ fixes a vertex of FF_N.

Another model: FS_N^* has $V(FS_N^*) = V(FS_N)$.
Two distinct vertices $\mathbb{A}, \mathbb{B}$ of FS_N^* are adjacent if there exists $w \in F_N, w \neq 1$ such that

$$||w||_{\mathbb{A}} = ||w||_{\mathbb{B}} = 0$$

i.e. w is conjugate to an elmt of a vertex group of $\mathbb{A}$ and w is conjugate to an elmnt of a vertex group of $\mathbb{B}$.

Open problems

Fact: For $N \geq 3$ the spaces FF_N and FS_N^* are quasi-isometric.

Yet another graph: The graph J_N has as its vertex set the set of (minimal nontrivial) splittings $F_N = \pi_1(\mathbb{A})$ such that $\mathbb{A}$ has one edge and a cyclic (trivial or $\mathbb{Z}$) edge group. Adjacency is again defined as having a common elliptic element.

Then FS_N^* is a subgraph of J_N and, moreover $V(FS_N^*)$ is a 4-dense subset of $V(J_N)$.

Problem 2. Is J_N Gromov-hyperbolic?

If $\phi \in Out(F_N)$ is a geometric iwip (comes from a pseudo-Anosov homeo of a compact surface with one bry component) then ϕ acts on J_N with a bounded orbit while ϕ acts as a hyperbolic isometry on FS_N^*.

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