New examples of totally disconnected locally compact groups

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A topological space X is

Hausdorff if for each $x \neq y$ there are disjoint open sets, one containing x and the other y

locally compact if for each x and each open set U containing x there is a compact open set V⊆U containing x

connected if it is not the disjoint union of two open sets

totally disconnected if for each $x \neq y$, X is the disjoint union of open sets, one containing x and the other y

G is a topological group if

G is a group and a topological space such that $(x, y) \mapsto xy^{-1}$ is a continuous map (from G×G to G)

Lem: Let G be a locally compact group and G_0 the **connected** component containing the identity. Then G_0 is an open normal subgroup and G/G_0 is **totally disconnected**.

In other words, to understand locally compact groups you just need to understand the *connected* and *totally disconnected* cases.

Understanding totally disconnected locally compact groups

Any (abstract) group G with the *discrete topology* is totally disconnected (and locally compact).

Question: What other (tdlc) topologies can you *put on* G?

Aut(Cay(G))

If G is finitely generated, let $\mathcal{T}$ be the topology on Aut(Cay(G)) with basis

$$\mathsf{N}(x, \mathsf{F}) = \{y \in \mathsf{Aut}(\mathsf{Cay}(\mathsf{G})) \mid x.f = y.f \quad \forall\ f \in \mathsf{F}\}$$

where F is a finite set of vertices of Cay(G).

Aut(Cay(G))

In some cases this topology is nondiscrete (*eg.* nonabelian free groups)

However, the subspace topology on G, or even the closure of G in Aut(Cay(G)), is discrete

(for each $\alpha \neq e \in$Aut(Cay(G)) there is some v so that $\alpha \notin$N$(e, \{v\})$ so the intersection of N$(e, \{v\})$ over all v is just $\{e\}$).

Instead, here is a trick with **commensurated subgroups** that sometimes makes a nondiscrete tdlc group in which G embeds densely.

Commensurability and commensurated subgroups

Defn: Let G be a group, and H, K subgroups. H and K are *commensurable* if $H \cap K$ is finite index in both H and K.

Lem: Commensurability is an equivalence relation

Commensurability and commensurated subgroups

Defn: H is *commensurated by G* if gHg^{-1} is commensurable with H for all $g \in$G.

Lem: If G is finitely generated, it suffices to check gHg^{-1} is commensurable with H just for the generators.

Example 1: Baumslag-Solitar groups

$\mathrm{BS}(m,n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$

the cyclic subgroup $\langle a \rangle$ is commensurated

Example 2: tdlc groups

Every tdlc group G has a **compact open subgroup** (van Dantzig).

An **automorphism** of a topological group $\alpha : G \to G$ is a group isomorphism that is also a homeomorphism (α and α^{-1} are continuous).

If V is a compact open subgroup of G, then $\alpha(V)$ is also compact and open, and $\alpha(V) \cap V$ is open, so its cosets in V are an open cover, its index is finite

(*i.e.* $\alpha(V) \cap V$ is commensurated by V)

Scale

Defn: $$s(\alpha) = \min_{\mathsf{V}\text{ compact open}} \{[\mathsf{V} : \alpha(\mathsf{V}) \cap \mathsf{V})\}$$

is the **scale** of the automorphism α.

A subgroup that realises this minimum for a group element is called **minimizing**.

Scale

In the case that α is the inner automorphism $x \mapsto gxg^{-1}$, the scale is a function s : G $\to \mathbb{Z}^+$

which satisfies some useful properties:

- s is continuous
- $s(x^n) = s(x)^n$
- $s(gxg^{-1}) = s(x)$
- the number of prime factors of the scales of a (compactly generated) tdlc group is finite

Recipe

Let G be an abstract group with a **commensurated** subgroup H, and suppose H has **no subgroup that is normal in G**.

Then G acts (faithfully) on G/H by permuting cosets, so G $\leq$ Sym(G/H).

if $x \notin$ H then xH$\neq$H

if $x \in$ H and xgH$=$ gH for all $g \in$G then $x \in \bigcap_{g \in G} g H g^{-1}$ which is normal so must be $\{e\}$

Recipe

Let $\mathcal{T}$ be the topology on Sym(G/H) with basis

$$\mathsf{N}(x, \mathsf{F}) = \{y \in \mathsf{Sym(G/H)} \mid y(g\mathsf{H}) = x(g\mathsf{H}) \ \forall \ (g\mathsf{H}) \in \mathsf{F}\}$$

for each $x \in$ Sym(G/H) and each finite subset F of G/H.

Recipe

Take the **closure** of G in Sym(G/H)

which is the intersection of all closed subsets of Sym(G/H) that contain G.

We denote the closed subgroup by G//H.

(G is dense in G//H)

Locally compact

Since H is commensurated, the orbits of cosets under H are finite,

$\mathsf{Stab}_{\mathsf{H}}(g\mathsf{H}) = \mathsf{N}(e, g\mathsf{H}) = \mathsf{H} \cap g\mathsf{H}g^{-1}$
so the orbit HgH is H/$\mathsf{Stab}_{\mathsf{H}}$ which is finite when H is commensurated

so H acts on G/H by permuting cosets in finite blocks,

so $\mathsf{H} \leq \prod \mathsf{Sym}(\mathsf{H}g\mathsf{H})$ which is compact by **Tychonov's theorem**.

The closure of H is also a subgroup of this compact group, so is **compact**. It is **open** since it is equal to $\mathsf{N}_{\mathsf{G}/\!\!/\mathsf{H}}(e,\mathsf{H})$.

It follows that G//H is locally compact since each point lies in a translate of $\overline{\mathsf{H}}$.

Totally disconnected

Since the action of G on G/H is faithful,

for each $x \neq y \in$ G there is a coset gH with xgH$\neq yg$H.

$\mathsf{N}_{\mathsf{G}/\!\!/\mathsf{H}}(x, g\mathsf{H})$ is an open set containing x, and its complement

$$\bigcup_{z \notin \mathsf{N}_{\mathsf{G}/\!\!/\mathsf{H}}(x,g\mathsf{H})} \mathsf{N}_{\mathsf{G}/\!\!/\mathsf{H}}(z, g\mathsf{H}) \text{ is open and contains } y.$$

So G//H is a tdlc group.

New examples

So given a group G, a subgroup H

- having no subgroups normal in G
- and commensurated by G

the recipe produces a ready-made tdlc group

Since $\langle a \rangle$ is commensurated by BS(m, n), and when $|m| \neq |n|$ has no subgroup that is normal in BS(m, n),

we get a (nondiscrete) topology on BS(m, n).

(*i.e.* we have a tdlc group in which BS(m, n) is dense)

Scales of BS$(m,n)//\langle a\rangle$

Thm (E, Willis): The set of scales for BS$(m,n)//\langle a\rangle$ for all $m,n\neq 0$ is

$$\left\{\left(\frac{\text{lcm}(m,n)}{m}\right)^k, \left(\frac{\text{lcm}(m,n)}{n}\right)^k \;:\; k\in\mathbb{N}\right\}$$

Since BS(m,n) is dense in its closure, and s: BS$(m,n)//\langle a\rangle\to\mathbb{Z}$ is continuous, if we show that scales of elements in BS(m,n) take only these values, the result for BS$(m,n)//\langle a\rangle$ follows.

See our paper (on arxiv very soon) for more details

Thanks and References

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