On the uniqueness of asymptotic cones of partially commutative groups

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Introduction

Trees

Bass-Serre theory: A group acts freely (without inversion of edges) by isometries on a tree if and only if it is a (subgroup of a) free group.

Real trees

Rips theorem (Bestvina-Feighn 1995, Gaboriau-Levitt-Paulin 1994): A finitely generated group acts freely on a real tree if and only if it is a free product of free abelian groups and (non-exceptional) surface groups.

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Trees —> Cubings

A tree is a contractible CW complex built up from edges (=1 cubes).
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Cubings —>Real cubings

What is a real cubing?
Real cubings are ultralimits of cubings

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Real trees —> Real cubings

C-Kazachkov (2011): A finitely generated group acts (essentially) freely (co-specially) on a real cubing if and only if it is a subgroup of a graph product of cyclic groups and (non-exceptional) surface groups.

Main results

Main Results

- Metric description of a real cubing
  (cf. Chiswell (1976), Tits (1977) , Alperin-Moss (1985): real trees)
- Existence and uniqueness of universal real cubings
  (cf. Mayer-Nikiel-Oversteegen (1992), Dyubina-Polterovich (2001))
- Uniqueness of asymptotic cones of pc groups

Partially commutative groups

Definition

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a (undirected) simplicial graph. The partially commutative group $\mathbb{G} = \mathbb{G}(\Gamma)$ defined by the commutation graph Γ is the group given by the following presentation,

$$\mathbb{G} = \langle V(\Gamma) \mid [v, v'] = 1, \text{ whenever } (v, v') \in E(\Gamma) \rangle.$$

- Introduced by Baudisch as semifree groups (1977);
- Graph groups (Droms, Servatius et al);
- Partially commutative groups (Duchamp, Krob et al);
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Examples

(On the blackboard)

PC groups and cubings

PC groups and cubings

The universal cover of (the Salvetti complex of) a pc group is a cubing.

Important aspects of the class of pc groups

Many important families of groups are subgroups of partially commutative groups.

Theorem (Agol, Kahn-Markovic, Wise 2012)

Let N be a non-positively curved (irreducible, closed) 3-manifold. Then the fundamental group $\pi_1(N)$ is virtually a subgroup of a partially commutative group.

Corollary

The manifold N is virtually Haken, virtually fibred and linear over $\mathbb{Z}$.

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Important aspects of the class of pc groups

- Many important counterexamples are given by subgroups of partially commutative groups:
  - Finiteness results (Bestvina-Brady 1997);
  - Undecidability problems (conjugacy, isomorphism, membership,...) Mikhailova for f.g (1966), Bridson-Wilton for f.p. (2012)
- Many problems in computer science can be formulated in terms of pc groups and pc monoids.
- Higher dimensional generalisation of free groups:

Slogan

The role played by free groups can be taken by pc groups: Rips' theory, model theory of groups, new generalisation of the class of hyperbolic groups.

Asymptotic cones

Asymptotic cones

- Let G be a finitely generated group.
- Let (X, d) be the Cayley graph of G (with the graph metric).
- Consider (X_i, d_i), where $X_i = X$ and $d_i = \frac{d}{n_i}$, $n_i < n_{i+1}$, $n_i \in \mathbb{N}$.

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$$d((g_i), (h_i)) = \lim_{i \to \infty} d_i(g_i, h_i)$$

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First problem: limit may not be well-defined.
Solution: take an ultrafilter ω in $\mathcal{P}(\mathbb{N})$ (a 0-1 finitely additive measure).
(Non-principal = measure 0 on finite sets).

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Second problem: d may be infinite.
Solution: fix a based point $(x_i) \in \prod_{i \in \mathbb{N}} (X_i, d_i)$ and consider only $V \subset \prod_{i \in \mathbb{N}} (X_i, d_i)$ at finite distance from (x_i).

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Last problem: different points are at distance 0.
Solution: quotient by the relation $(g_i) \sim (h_i)$ if and only if $d((g_i), (h_i)) = 0$.

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The asymptotic cones of G:

$$Asy(G, \{n_i\}, (x_i)) = V / \sim .$$

where $V \subset \prod_{i \in \mathbb{N}} (X_i, d_i)$ so that $d((x_i), (v_i)) < \infty$, $(v_i) \in V$ and $(g_i) \sim (h_i)$ if and only if $d((g_i), (h_i)) = 0$.

Examples

- Asy($\mathbb{Z}$)=$\mathbb{R}$
- Asy(free group)=real tree —0-hyperbolic metric space—
- Asy(hyperbolic group)=real tree.
- Asy(pc group)=real cubing
- There exist finitely generated and finitely presented groups with different asymptotic cones (Thomas-Velickovic 2000, Drutu-Sapir 2005, Olshanski-Sapir 2007, Osin-Ould-Houcine 2011)

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Asymptotic cones of pc groups

Question

How do asymptotic cones of pc groups look like?

Vicinities and the geometry of real cubings

- Asymptotic cones of partially commutative groups are homogeneous.
- Can we understand the global geometry of the space from the local geometry?
- Obvious choice: balls.
- Balls are too big.

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Vicinities and the geometry of real cubings

Universal real tree: tree for which the valency at each point is uncountable.

Definition (Vicinity)

Let X be a CAT(0) metric space. A vicinity V_p of a point $p \in X$ is a subspace of X such that:

- $\forall x \in V_p$, the geodesic $[p, x] \subset V_p$,
- $\forall x, y \in V_p$, either $[p, x]$ and $[p, y]$ intersect only in $\{p\}$ or one contains the other one,
- $\forall x \in X$, $\{p\} \subsetneq [p, x] \cap V_p$.

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- Euclidean plane
- Tree
- Universal cover of a pc group.

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Real cubings: local geometry

Definition

A 1-cubing is a cubing such that all cubes intersect in a point.
A u-cubing is a connected space constructed from a union of Euclidean spaces (with a system of coordinates) by identifying subspaces obtained from the projection of some of the coordinates, i.e. a maximal standard vicinity of a pc group.

Definition (Metric description of real cubings)

A CAT(0) metric space X is a (universal) real cubing if for all $x \in X$ there exists a vicinity V_x convex and isometric to a 1-cubing (to a u-cubing).

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Real cubings: global geometry

Theorem (C., Kazachkov, Sisto)

Let V be a u-cubing. Then there exists a unique CAT(0) metric space X for which the vicinity of each point is convex and isometric to V.

Remark

In the particular case when X is an uncountable sheaf of lines, we recover the uniqueness of universal real trees (Mayer-Nikiel-Oversteegen (1992), Dyubina-Polterovich (2001)).

Corollary

A CAT(0) metric space X is a real cubing if and only if X is a convex subspace of a universal real cubing. There exists a unique universal real cubing containing all real cubings.

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Asymptotic cones of pc groups

Asymptotic cones of pc groups: step 1

The asymptotic cone of a pc group $Asy(G, (n_i), (id))$ is a universal real cubing.

Asymptotic cones of pc groups: step 2

The vicinity of the asymptotic cone of a pc group $Asy(G, (n_i), (id))$ is independent of the rescaling sequence (and choice of ultrafilter).

Theorem (C., Kazachkov, Sisto)

Given a partially commutative group G all of its asymptotic cones are isometric.

Outlook: New class $\mathcal{C}$

Hyperbolic groups

A f.g. group is hyperbolic if and only if its asymptotic cones is unique and it is a universal real tree.

New class $\mathcal{C}$

A f.g. group is in the class $\mathcal{C}$ if and only if its asymptotic cone is unique and it is a universal real cubing (and *)

Outlook: New class $\mathcal{C}$

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Example (Groups in the class $\mathcal{C}$)

- Partially commutative groups;
- Hyperbolic groups;
- Limit groups - asymptotic cone is tree-graded with respect to Euclidean spaces-;
- Relatively hyperbolic groups relative to $G_i \in \mathcal{C}$.

Outlook: New class $\mathcal{C}$

New class $\mathcal{C}$

A f.g. group is in the class $\mathcal{C}$ if and only if its asymptotic cone is unique and it is a universal real cubing (and *)

Are the following groups in the class $\mathcal{C}$?

- Mapping class groups?
- $Out(F_n)$?

Outlook: New class $\mathcal{C}$

New class $\mathcal{C}$

A f.g. group is in the class $\mathcal{C}$ if and only if its asymptotic cone is unique and it is a universal real cubing (and *)

Example (Properties of the class $\mathcal{C}$)

- The class $\mathcal{C}$ is closed under taking direct products, free products and, more generally, graph products.
- Is the class closed under taking limits: if H is a limit group over G, $G \in \mathcal{H}$, is $H < G'$ where $G' \in \mathcal{C}$?

Outlook: New class $\mathcal{C}$

Class $\mathcal{C}$

Provides a uniform way to study these groups.

Outlook: New class $\mathcal{C}$

Develop a cancellation/Dehn filling theory

- Let F be a free group and let $w_1, \dots, w_k \in F$. Then the group $\langle F \mid w_1^N, \dots, w_k^N \rangle$ is hyperbolic, for N sufficiently large.
- Let G be a pc group and let $w_1, \dots, w_k \in G$ be irreducible. Then the group $\langle G \mid w_1^N, \dots, w_k^N \rangle$ is in $\mathcal{C}$, for N sufficiently large.

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Outlook: New class $\mathcal{C}$

Theorem (C., Kazachkov, 2011)

Description of limit groups over partially commutative groups via actions on real cubings.

Develop a structure theory for groups acting on real cubings

- Description of limit groups over any group from the class $\mathcal{C}$.
- Apply it to the study of automorphisms of pc groups: $Out(G)$.

THANK YOU!