Grigorchuk's group

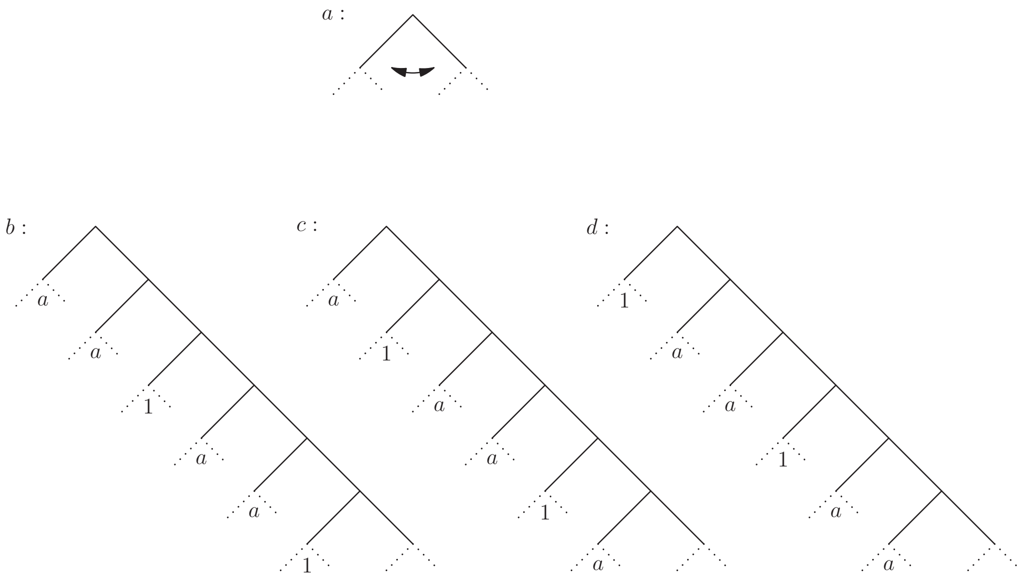

Fig. 1. Grigorchuk's group $G = \langle a, b, c, d\rangle$ acts by automorphisms on the rooted binary tree T. The generators of G satisfy the following recurrent conditions:

$$b = (a, c), \quad , c = (a, d), \quad d = (1, b).$$

Note that $a^2 = b^2 = c^2 = d^2 = 1$ and that $\langle b, c, d\rangle$ is the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Below we expose a proof of the following theorem.
Theorem (Grigorchuk). The group G has a subexponential growth.

1 Definitions and notations

Let $\mathrm{St}(n)$ be the subgroup of G consisting of all automorphisms which fix each vertex of the level n. It is easy to check that $\mathrm{St}(1) = \langle b, c, d, aba, aca, ada\rangle$.

Let S^* be the free monoid generated by the set $S = \{a, b, c, d\}$. There is a natural surjective homomorphism $S^* \to G$.

Let S^*_{even} be the submonoid of S consisting of all words with even number of occurrences of a. Then there is a natural surjective homomorphism $S^*_{even} \to \mathrm{St}(1)$.

We define a homomorphism $\varphi = (\varphi_0, \varphi_1) : S^*_{even} \to S^* \times S^*$ by the formulas

$$\begin{array}{ll} \varphi(b) = (a, c), & \varphi(aba) = (c, a) \\ \varphi(c) = (a, d), & \varphi(aca) = (d, a) \\ \varphi(d) = (1, b), & \varphi(ada) = (b, 1). \end{array} \qquad (i)$$

Any word $w \in S^*$ can be reduced by applying a finite number of the following elementary reductions:

Type 1: $bc \rightsquigarrow d$, $cb \rightsquigarrow d$, $bd \rightsquigarrow c$, $db \rightsquigarrow c$, $cd \rightsquigarrow b$, $dc \rightsquigarrow b$.
Type 2: $a^2 \rightsquigarrow 1$, $b^2 \rightsquigarrow 1$, $c^2 \rightsquigarrow 1$, $d^2 \rightsquigarrow 1$.

Example. $abcdad \rightsquigarrow addad \rightsquigarrow aad \rightsquigarrow d$.

Reduced words have one of the following forms, where $u_i \in \{b, c, d\}$ for all i:

$$w = \begin{cases} au_1au_2 \cdot au_3au_4 \cdot \ldots \cdot au_{2m-1}au_{2m}, \\ au_1au_2 \cdot au_3au_4 \cdot \ldots \cdot au_{2m-1}a, \\ u_0au_1au_2 \cdot au_3au_4 \cdot \ldots \cdot au_{2m-1}a, \\ u_0au_1au_2 \cdot au_3au_4 \cdot \ldots \cdot au_{2m-1}au_{2m}. \end{cases} \qquad (ii)$$

The length of w is denoted by $|w|$. The number of occurrences of a in w is denoted by $|w|_a$. Analogously, the number of occurrences of b and c in w is denoted by $|w|_{b,c}$, and so on.

Claim 1. For any reduced word $w \in S^*$ holds

$$\frac{|w|-1}{2} \leqslant |w|_{b,c,d} \leqslant \frac{|w|+1}{2}. \qquad (iii)$$

Let $w^\bullet$ be the reduced form of w. For $i = 1, 2$, let $r_i(w)$ be the numbers of elementary reductions of type i in the process of reduction $w \rightsquigarrow w^\bullet$. One can prove that the word $w^\bullet$ and the numbers $r_1(w)$ and $r_2(w)$ do not depend on a choice of reduction process. The following number is called the number of *weighted reductions* for w.

$$\rho(w) = \tau_1(w) + 2\tau_2(w)$$

We also denote

$$\rho_1(w) = \rho(\varphi_0(w)) + \rho(\varphi_1(w)).$$

Claim 2. For any word $w \in S^*$, we have

$$|w^\bullet|_d \geqslant |w|_d - \rho(w), \qquad (iv)$$

$$|w^\bullet|_{c,d} \geqslant |w|_{c,d} - 2\rho(w). \qquad (v)$$

Proof. The factor 2 in (v) is because the reduction $cd \rightsquigarrow b$ decreases the number of c and d in w by 2. □

For $i = 0, 1$, we denote $\varphi_i^\bullet(w) = (\varphi_i(w))^\bullet$.

Claim 3. For any reduced word $w \in S^*$, we have

$$|\varphi_0(w)|_d + |\varphi_1(w)|_d = |w|_c. \qquad (vi)$$

$$|\varphi_0^\bullet(w)|_d + |\varphi_1^\bullet(w)|_d \geqslant |w|_c - 2\rho_1(w). \qquad (vii)$$

Proof. (vi) follows from (i); (vii) follows from (iv) and (vi).

2 The main lemma

Lemma 1. Let w be a reduced word from S^* which corresponds to an automorphism $g \in \mathrm{St}(3)$. Then

$$\sum_{i,j,k\in\{0,1\}} |\varphi_i^\bullet(\varphi_j^\bullet(\varphi_k^\bullet(w)))| \leqslant \frac{3}{4}|w| + 8. \qquad (\star)$$

Proof. First we prove Claims 1-5 below.

`Step 1.`

Claim 1.

$$|\varphi_0(w)| + |\varphi_1(w)| \leqslant |w| + 1 - |w|_d \qquad (1)$$

$$|\varphi_0^\bullet(w)| + |\varphi_1^\bullet(w)| \leqslant |w| + 1 - |w|_d - \rho_1, \qquad (1^\bullet)$$

Proof. The inequality (1) without the term 1 on the right side holds for all reduced words of length 4 which begin with a:

$$\begin{aligned}
\varphi(abab) &= (ca, ac)\\
\varphi(abac) &= (ca, ad)\\
\varphi(abad) &= (c1, ab)\\
\varphi(acab) &= (da, ac)\\
\varphi(acac) &= (da, ad)\\
\varphi(acad) &= (d1, ab)\\
\varphi(adab) &= (ba, 1c)\\
\varphi(adac) &= (ba, 1d)\\
\varphi(adad) &= (b1, 1b)
\end{aligned}$$

In the general case, (1) can be verified with the help of (ii). □

Claim 2.

$$|\varphi_0(w)|_{c,d} + |\varphi_1(w)|_{c,d} = |w|_{b,c}. \qquad (2)$$

$$\begin{aligned}
|\varphi_0^\bullet(w)|_{c,d} + |\varphi_1^\bullet(w)|_{c,d} &\geqslant |w|_{b,c} - 2\rho_1.\\
&\geqslant \frac{|w|-1}{2} - |w|_d - 2\rho_1
\end{aligned} \qquad (2^\bullet)$$

Proof. (2) follows from (i); $(2^\bullet)$ follows from (2) and (v) and (iii). □

`Step 2.` We set

$$\begin{aligned}
\rho_2 &= \rho(\varphi_0(\varphi_0^\bullet(w))) + \rho(\varphi_1(\varphi_0^\bullet(w))) + \rho(\varphi_0(\varphi_1^\bullet(w))) + \rho(\varphi_1(\varphi_1^\bullet(w)))\\
&= \rho_1(\varphi_0^\bullet(w)) + \rho_1(\varphi_1^\bullet(w)).
\end{aligned}$$

Claim 3.

$$\sum_{i,j\in\{0,1\}} |\varphi_i^\bullet(\varphi_j^\bullet(w))| \leqslant |w| + 3 - |w|_d - \rho_1 - |\varphi_0^\bullet(w)|_d - |\varphi_1^\bullet(w)|_d - \rho_2. \qquad (3)$$

Proof. We denote by L and R the left and the right sides of the inequality, respectively. Applying ($1^\bullet$) twice, we obtain

$$\begin{aligned} L &= \sum_{j=0}^{1} |\varphi_0^\bullet(\varphi_j^\bullet(w))| + |\varphi_1^\bullet(\varphi_j^\bullet(w))| \\ &\leqslant \sum_{j=0}^{1} \Big(|\varphi_j^\bullet(w)| + 1 - |\varphi_j^\bullet(w)|_d - \rho_1(\varphi_j^\bullet(w)) \Big) \leqslant R. \end{aligned}$$

□

Claim 4.

$$\sum_{i,j\in\{0,1\}} |\varphi_i^\bullet(\varphi_j^\bullet(w))|_d \geqslant \frac{|w|-1}{2} - |w|_d - 2\rho_1 - |\varphi_0^\bullet(w)|_d - |\varphi_1^\bullet(w)|_d - 2\rho_2. \tag{4}$$

Proof. We denote by L and R the left and the right sides of the inequality, respectively. Applying (vii) and ($2^\bullet$), we obtain

$$\begin{aligned} L &= \sum_{j=0}^{1} |\varphi_0^\bullet(\varphi_j^\bullet(w))|_d + |\varphi_1^\bullet(\varphi_j^\bullet(w))|_d \\ &\geqslant \sum_{j=0}^{1} (|\varphi_j^\bullet(w)|_c - 2\rho_1(\varphi_j^\bullet(w))) = |\varphi_0^\bullet(w)|_c + |\varphi_1^\bullet(w)|_c - 2\rho_2 \\ &= |\varphi_0^\bullet(w)|_{c,d} + |\varphi_1^\bullet(w)|_{c,d} - |\varphi_0^\bullet(w)|_d - |\varphi_1^\bullet(w)|_d - 2\rho_2 \geqslant R. \end{aligned}$$

`Step 3.`

Claim 5.

$$\sum_{i,j,k\in\{0,1\}} |\varphi_i^\bullet(\varphi_j^\bullet(\varphi_k^\bullet(w)))| \leqslant \frac{|w|}{2} + 8 + \rho_1 + \rho_2. \tag{5}$$

Proof. We denote by L and R the left and the right sides of the inequality, respectively. Then

$$\begin{aligned} L &= \sum_{j,k\in\{0,1\}} |\varphi_0^\bullet(\varphi_j^\bullet(\varphi_k^\bullet(w)))| + |\varphi_1^\bullet(\varphi_j^\bullet(\varphi_k^\bullet(w)))| \\ &\overset{(1)}{\leqslant} \sum_{j,k\in\{0,1\}} |\varphi_j^\bullet(\varphi_k^\bullet(w))| + 1 - |\varphi_j^\bullet(\varphi_k^\bullet(w))|_d \\ &\overset{(3),(4)}{\leqslant} \Big(|w| + (3+4) - |w|_d - \rho_1 - |\varphi_0^\bullet(w)|_d - |\varphi_1^\bullet(w)|_d - \rho_2 \Big) \\ &\quad -\Big(\tfrac{|w|-1}{2} - |w|_d - 2\rho_1 - |\varphi_0^\bullet(w)|_d - |\varphi_1^\bullet(w)|_d - 2\rho_2 \Big) \leqslant R. \end{aligned}$$

□

Now we are ready to finish the proof of Lemma 1.

If $\rho_1 + \rho_2 \leqslant \frac{|w|}{4}$, then ($\star$) follows from (5).

If $\rho_1 + \rho_2 > \frac{|w|}{4}$, then we have from (3) that

$$\sum_{j,k\in\{0,1\}} |\varphi_j^\bullet(\varphi_k^\bullet(w))| \leqslant \frac{3}{4}|w| + 3. \tag{6}$$

Then

$$\begin{aligned}\sum_{i,j,k\in\{0,1\}} |\varphi_i^\bullet(\varphi_j^\bullet(\varphi_k^\bullet(w)))| &= \sum_{j,k\in\{0,1\}} |\varphi_0^\bullet(\varphi_j^\bullet(\varphi_k^\bullet(w)))| + |\varphi_1^\bullet(\varphi_j^\bullet(\varphi_k^\bullet(w)))| \\ &\overset{(1)}{\leqslant} \sum_{j,k\in\{0,1\}} |\varphi_j^\bullet(\varphi_k^\bullet(w))| + 1 - |\varphi_j^\bullet(\varphi_k^\bullet(w))|_d \\ &\overset{(6)}{\leqslant} \tfrac{3}{4}|w| + 7.\end{aligned}$$

□

Definition. For $g \in \mathrm{St}(3)$ and $i, j, k \in \{0,1\}$ let $g_{i,j,k}$ be the automorphism induced by g on the subtree of T with the root i, j, k. We can consider $g_{i,j,k}$ as an element of G. By $\ell_S(g)$ we denote the length of g with respect to $S = \{a, b, c, d\}$.

Corollary 1. For $g \in \mathrm{St}(3)$ holds

$$\sum_{i,j,k\in\{0,1\}} \ell_S(g_{i,j,k}) \leqslant \frac{3}{4}\ell_S(g) + 8.$$

3 The group G has a subexponential growth

Lemma 2. Let G be a group generated by a finite set S. Suppose that G_0 is a subgroup of finite index m in G. We consider the growth function of G with respect to S,

$$\beta(k) = |\{g \in G \mid \ell_S(g) \leqslant k\}|$$

and the relative growth function of G_0 with respect to S,

$$\beta_0(k) = |\{g \in G_0 \mid \ell_S(g) \leqslant k\}|.$$

Then

$$\beta(k) \leqslant m\beta_0(k + m - 1).$$

Theorem (Grigorchuk). The group G has a subexponential growth.

Proof. For $G_0 = \mathrm{St}(3)$ we have $m = |G : G_0| = 2^7$. We denote

$$\omega = \lim_{n\to\infty} \beta(n)^{1/n}.$$

It suffices to show that $\omega = 1$. Let $\epsilon > 0$. Then there exists $C > 0$ such that

$$\beta(n) \leqslant C \cdot (\omega + \epsilon)^n$$

for every $n \in \mathbb{N}$. Note that any $g \in \mathrm{St}(3)$ is completely determined by the induced automorphisms $g_{i,j,k}$, where i, j, k run over $\{0,1\}$. From Corollary 1 we deduce

$$\begin{aligned}\beta_0(n) &\leqslant \sum_{n_1+\cdots+n_8\leqslant\frac{3}{4}n+8} \beta(n_1)\beta(n_2)\ldots\beta(n_8) \leqslant C^8(\omega+\epsilon)^{\frac{3}{4}n+8} \sum_{n_1+\cdots+n_8\leqslant\frac{3}{4}n+8} 1 \\ &\leqslant C^8(\omega+\epsilon)^{\frac{3}{4}n+8}P(n),\end{aligned} \tag{!}$$

where $P(n)$ is a polynomial of degree 9. By Lemma 2, we have $\beta(n) \leqslant 2^7 \cdot \beta_0(n + 2^7 - 1)$. Using (!), we deduce

$$\omega = \lim_{n\to\infty} \beta(n)^{1/n} = (\omega + \epsilon)^{\frac{3}{4}}.$$

Since this holds for any $\epsilon > 0$, we have $\omega \leqslant \omega^{\frac{3}{4}}$, hence $\omega \leqslant 1$. □