# Problems for Tutorial 5

(Thursday, 16.12, at 10 a.m.)

**Problem 1.**

(1) Let $G_1$ be a topological group, $G_2$ be a group and $\varphi : G_1 \to G_2$ an epimorphism. We consider $G_2$ as a topological group with respect to the quotient topology. Let $H$ be a subgroup of $G_2$. Prove the following statements.

   (a) If $\varphi^{-1}(H)$ is discrete in $G_1$, then $H$ is discrete in $G_2$. [4 P.]

   (b) Suppose additionally that $G_1$ is Hausdorff and that $\ker(\varphi)$ is finite. If $H$ is discrete in $G_2$, then $\varphi^{-1}(H)$ is discrete in $G_1$. [4 P.]

(2) Deduce from this that a subgroup $H$ of $\mathrm{PSL}_2(\mathbb{R})$ is discrete if and only if its full preimage $\varphi^{-1}(H)$ is discrete in $\mathrm{SL}_2(\mathbb{R})$. [2 P.]

**Problem 2.** Consider the set $\widehat{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$, where $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. Define a topology $\mathcal{T}$ on $\widehat{\mathbb{H}}$ so that the following three properties are satisfied. [9 P.]

(a) The usual topologies on $\mathbb{H}$ and on $\mathbb{R}$ are induced by the topology on $\widehat{\mathbb{H}}$.

(b) The closure of $\mathbb{H}$ in $\widehat{\mathbb{H}}$ is $\widehat{\mathbb{H}}$.

(c) $(\widehat{\mathbb{H}}, \mathcal{T})$ is a compact topological space.

**Problem 3.** Recall that for any subset $S \subseteq \mathbb{H}$ we denote by $\mathbf{AP}_{\widehat{\mathbb{H}}}(S)$ the set of accumulation points of $S$ in $\widehat{\mathbb{H}}$. Recall that the *limit set* of a subgroup $G \leqslant \mathrm{PSL}_2(\mathbb{R})$ is defined to be as

$$\Lambda(G) = \bigcup_{z \in \mathbb{H}} \mathbf{AP}_{\widehat{\mathbb{H}}}(G(z)).$$

(a) Prove that for any point $z_0 \in \mathbb{H}$ we have [4 P.]

$$\Lambda(G) = \mathbf{AP}_{\widehat{\mathbb{H}}}(G(z_0)).$$

(b) Prove that if $G$ is a Fuchsian group, then $\Lambda(G)$ is closed. [4 P.]

(c) Let $G = \mathrm{PSL}_2(\mathbb{R})$. Prove that $\Lambda(G) = \mathbb{R} \cup \{\infty\}$. [4 P.]