Problems for Tutorial 3

(Thursday, 2.12, at 10 a.m.)

Problem 1.

(1) Let G be a topological group. Let N be a normal subgroup of G.

  (a) Prove that G/N is a topological group with respect to the quotient topology. [5 P.]

  (b) Prove that the topology on G/N is Hausdorff if and only if N is closed in G. [5 P.]

(2) Prove that $\mathrm{PSL}_2(\mathbb{R})$ is a topological group with respect to the topology defined on the last lecture. [2 P.]

Problem 2. Consider

$$A = \begin{pmatrix} 5 & -3 \\ \frac{9}{2} & -\frac{5}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

For $X \in \{A, B\}$ definie the type of the Möbius transformation T_X, compute $\widehat{\mathrm{Fix}}(T_X)$ and find all T_X-invariant geodesic lines in $\mathbb{H}$. [5 P.]

Problem 3.

(a) Let $\varphi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ be the canonical epimorphism. Prove that a subgroup H of $\mathrm{PSL}_2(\mathbb{R})$ is discrete if and only if its full preimage $\varphi^{-1}(H)$ is discrete in $\mathrm{SL}_2(\mathbb{R})$. [5 P.]

(b) Prove that each Fuchsian group is countable. [5 P.]

Problem 4. Let $r > 0$. Consider the Möbius transformations

$$\theta_r : \quad z \to rz,$$
$$\psi : \quad z \mapsto -\frac{1}{z}.$$

Prove that the subgroup $H_r := \langle \theta_r, \psi \rangle$ is discrete in $\mathrm{M\ddot{o}b}_{\mathbb{R}}$. [5 P.]

Problem 5. Prove that each discrete subgroup G of $\mathbb{R}^n$ is finitely generated[1]. [5 P.]

Hint. We consider the vector subspace

$$U_G := \{r_1 g_1 + \cdots + r_s g_s \mid s \in \mathbb{N}, r_i \in \mathbb{R}, g_i \in G, i = 1, \ldots, s\}.$$

Let $v_1, \ldots, v_m \in G$ be its $\mathbb{R}$-basis. (Why such a basis exists?) Consider the subgroup $H := \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_m$ of G. Prove that the group G/H is finite. Deduce from this that G is finitely generated.

Please read the second page.

[1] From the classification of finitely generated abelian groups, it follows that $G \cong \mathbb{Z}^k$ for some $k \in \mathbb{N} \cup \{0\}$.

`Definition.` Let X be a topological space.

(a) A subset $Y \subseteq X$ is called *dense* in X if for every $x \in X$ every neighborhood of x contains an element from Y.

(b) The topological space X is called *separable* if X has a dense countable subset[a].

(c) The topological space X is called *second countable* if there exists a countable set of open subsets $U_1, U_2, \ldots$ so that for every $x \in X$ and every neighborhood U of x, there exists U_i with $x \in U_i \subseteq U$.

[a] In particular, $\mathbb{R}$ with respect to the usual topology is separable.

Problem 5. Prove that every separable metric space (X, d) is second countable as a topological space. [5 P.]

Problem 6. We define a topology on $\mathbb{R}$ which differs from the classical one: The basis of this topology consists of all sets of the form $[a, b)$ $(a < b)$. Let $(\mathbb{R}, \mathcal{T})$ be the resulting topological space. Prove that [5 P.]

(a) $(\mathbb{R}, \mathcal{T})$ is separable; [2 P.]

(b) $(\mathbb{R}, \mathcal{T})$ is not second countable; [3 P.]

(b) $(\mathbb{R}, \mathcal{T})$ is not metrizable. [1 P.]