Solutions to Bestvina & Feighn's Exercises on Limit Groups

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Abstract

This article gives solutions to the exercises in Bestvina and Feighn's paper [2] on Sela's work on limit groups. We prove that all constructible limit groups are limit groups and give an account of the shortening argument of Rips and Sela.

Mladen Bestvina and Mark Feighn's beautiful first set of notes [2] on Zlil Sela's work on the Tarski problems (see [10] *et seq.*) provides a very useful introduction to the subject. It gives a clear description of the construction of Makanin–Razborov diagrams, and precisely codifies the structure theory for limit groups in terms of *constructible limit groups (CLGs)*. Furthermore, the reader is given a practical initiation in the subject with exercises that illustrate the key arguments. This article is intended as a supplement to [2], to provide solutions to these exercises. Although we do give some definitions in order not to interrupt the flow, we refer the reader to [2] for all the longer definitions and background ideas and references.

1 Definitions and elementary properties

In this section we present solutions to exercises 2, 3, 4, 5, 6 and 7, which give some of the simpler properties and the first examples and non-examples of limit groups.

1.1 ω-residually free groups

Fix $\mathbb{F}$ a free group of rank $r > 1$.

Definition 1.1 *A finitely generated group G is* ω-residually free *if, for any finite subset $X \subset G$, there exists a homomorphism $h : G \to \mathbb{F}$ whose restriction to X is injective. (Equivalently, whenever $1 \notin X$ there exists a homomorphism $h : G \to \mathbb{F}$ so that $1 \notin h(X)$.)*

Residually free groups inherit many of the properties of free groups; the first and most obvious property is being torsion-free.

Lemma 1.2 (Exercise 2 of [2]) *Any residually free group is torsion-free.*

Proof. Let G be ω-residually free (indeed G can be thought of as merely residually free). Then for any $g \in G$, there exists a homomorphism $h : G \to \mathbb{F}$ with $h(g) \neq 1$; so $h(g^k) \neq 1$ for all integers k, and $g^k \neq 1$. $\square$

It is immediate that any subgroup of an ω-residually free group is ω-residually free (exercise 6 of [2]).

That the choice of $\mathbb{F}$ does not matter follows from the observation that all finitely generated free groups are ω-residually free.

Example 1.3 (Free groups) *Let F be a finitely generated free group. Realize $\mathbb{F}$ as the fundamental group of a rose Γ with r petals; that is, the wedge of r circles. Then Γ has an infinite-sheeted cover that corresponds to a subgroup F' of $\mathbb{F}$ of countably infinite rank. The group F can be realized as a free factor of F'; this exhibits an injection $F \hookrightarrow \mathbb{F}$. In particular, every free group is ω-residually free.*

Example 1.4 (Free abelian groups) *Let A be a finitely generated free abelian group, and let $a_1, \ldots, a_n \in A$ be non-trivial. Fix a basis for A, and consider the corresponding inner product. Let $z \in A$ be such that $\langle z, a_i \rangle \neq 0$ for all i. Then inner product with z defines a homomorphism $A \to \mathbb{Z}$ so that the image of every a_i is non-trivial, as required.*

Examples 1.3 and 1.4 give exercise 3 of [2].

1.2 Limit groups

Groups that are ω-residually free are natural examples of limit groups.

Definition 1.5 *Let $\mathbb{F}$ be as above, and Γ a finitely generated group. A sequence of homomorphisms $(f_n : \Gamma \to \mathbb{F})$ is* stable *if, for every $g \in G$, $f_n(g)$ is either eventually* 1 *or eventually not* 1*. The* stable kernel *of a stable sequence of homomorphisms (f_n) consists of all $g \in G$ with $f_n(g)$ eventually trivial; it is denoted $\underrightarrow{ker} f_n$.*

A limit group *is a group arising as a quotient $\Gamma / \underrightarrow{ker} f_n$ for (f_n) a stable sequence.*

Lemma 1.6 (Exercise 5 of [2]) *Every ω-residually free group is a limit group.*

Proof. Let G be an ω-residually free group. Fix a generating set, and let $X_n \subset G$ be the ball of radius n about the identity in the word metric. Let $f_n : G \to \mathbb{F}$ be a homomorphism that is injective on X_n. Now f_n is a stable sequence and the stable kernel is trivial, so G is a limit group. $\square$

In fact every limit group is ω-residually free (lemma 1.11 of [2]). Henceforth, we shall use the terms interchangeably.

1.3 Negative examples

Let's see some examples of groups that aren't limit groups. The first three examples are surface groups that aren't even residually free. It follows from lemma 1.2 that the fundamental group of the real projective plane is not a limit group. A slightly finer analysis yields some other negative examples.

Lemma 1.7 *The only 2-generator residually free groups are the free group of rank 2 and the free abelian group of rank 2.*

Proof. Let G be a residually free group generated by x and y. If G is non-abelian then $[x, y] \neq 1$ so there exists a homomorphism $f : G \to \mathbb{F}$ with $f([x, y]) \neq 1$. So $f(x)$ and $f(y)$ generate a rank 2 free subgroup of $\mathbb{F}$. Therefore G is free. $\square$

In particular, the fundamental group of the Klein bottle is not a limit group. Only one other surface group fails to be ω-residually free. This was first shown by R. S. Lyndon in [5].

Lemma 1.8 (The surface of Euler characteristic -1) *Let Σ be the closed surface of Euler characteristic -1. Then any homomorphism $f_* : \pi_1(\Sigma) \to \mathbb{F}$ has abelian image. In particular, since $\pi_1(\Sigma)$ is not abelian, it is not residually free.*

Proof. Let Γ be a bouquet of circles so $\mathbb{F} = \pi_1(\Gamma)$. Realize the homomorphism f_* as a map from Σ to Γ, which we denote by f. Our first aim is to find an essential simple closed curve in the kernel of f_*.

Consider x the mid-point of an edge of Γ. Altering f by a homotopy, it can be assumed that f is transverse at x; in this case, $f^{-1}(x)$ is a collection of simple closed curves. Let γ be such a curve. If γ is null-homotopic in Σ then γ can be removed from $f^{-1}(x)$ by a homotopy. If all components of the pre-images of all midpoints x can be removed in this way then f_* was the trivial homomorphism. Otherwise, any remaining such component γ lies in the kernel of f as required.

We proceed with a case-by-case analysis of the components of $\Sigma \smallsetminus \gamma$.

1. If γ is 2-sided and separating then, by examining Euler characteristic, the components of $\Sigma \smallsetminus \gamma$ are a punctured torus or a Klein bottle, together with a Möbius band. So f factors through the one-point union $T \vee \mathbb{R}P^2$ or $K \vee \mathbb{R}P^2$. In either case, it follows that the image is abelian.

2. If γ is 2-sided and non-separating then $\Sigma - \gamma$ is the non-orientable surface with Euler characteristic -1 and two boundary components, so f_* factors through $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and hence through $\mathbb{Z}$.

3. If γ is 1-sided then γ^2 is 2-sided and separating, and case 1 applies.

This finishes the proof. $\square$

Lemmas 1.2, 1.7 and 1.8 give exercise 4 of [2]. Here is a more interesting obstruction to being a limit group. A group G is *commutative transitive* if every non-trivial element has abelian centralizer; equivalently, if $[x, y] = [y, z] = 1$ then $[x, z] = 1$. Note that $\mathbb{F}$ is commutative transitive, since every non-trivial element has cyclic centralizer.

Lemma 1.9 (Exercise 7 of [2]) *Limit groups are commutative transitive.*

Proof. Let G be ω-residually free, let $g \in G$, and suppose $a, b \in G$ commute with g. Then there exists a homomorphism

$$f : G \to \mathbb{F}$$

injective on the set $\{1, g, [a, b]\}$. Then $f([g, a]) = f([g, b]) = 1$; since $\mathbb{F}$ is commutative transitive, it follows that

$$f([a, b]) = 1.$$

So $[a, b] = 1$, as required. $\square$

A stronger property also holds. A subgroup $H \subset G$ is *malnormal* if, whenever $g \notin H$, $gHg^{-1} \cap H = 1$. The group G is *CSA* if every maximal abelian subgroup is malnormal.

Remark 1.10 *If G is CSA then G is commutative transitive. For, let $g \in G$ with centralizer $Z(g)$. Consider maximal abelian $A \subset Z(g)$ and $h \in Z(g)$. Then $g \in hAh^{-1} \cap A$, so $h \in A$. Therefore $Z(g) = A$.*

Lemma 1.11 *Limit groups are CSA.*

Proof. Let $H \subset G$ be a maximal abelian subgroup, consider $g \in G$, and suppose there exists non-trivial $h \in gHg^{-1} \cap H$. Let $f : G \to \mathbb{F}$ be injective on the set

$$\{1, g, h, [g, h]\}.$$

Then $f([h, ghg^{-1}]) = 1$, which implies that $f(h)$ and $f(ghg^{-1})$ lie in the same cyclic subgroup. But in a free group, this is only possible if $f(g)$ also lies in that cyclic subgroup; so $f([g, h]) = 1$, and hence $[g, h] = 1$. By lemma 1.9 it follows that g commutes with every element of H, so $g \in H$. $\square$

2 Embeddings in real algebraic groups

In this section we provide solutions to exercises 8 and 9 of [2], which show how to embed limit groups in real algebraic groups and also $PSL_2(\mathbb{R})$, and furthermore give some control over the nature of the embeddings. First, we need a little real algebraic geometry.

By, for example, proposition 3.3.13 of [3], every real algebraic variety V has an open dense subset $V_{\text{reg}} \subset V$ with finitely many connected components, so that every component $V' \subset V_{\text{reg}}$ is a manifold.

Lemma 2.1 *Consider a countable collection $V_1, V_2, \ldots \subset V$ of closed subvarieties. Then for any component V' of V_{reg} as above, either there exists k so that $V' \subset V_k$, or*

$$V' \cap \bigcup_{i=1}^{\infty} V_i$$

has empty interior.

Proof. Suppose $V' \cap \bigcup_i V_i$ doesn't have empty interior. Then, by Baire's Category Theorem, there exists k such that $V' \cap V_k$ doesn't have empty interior. Consider x in the closure of the interior of $V' \cap V_k$ and let f be an algebraic function on V' that vanishes on V_k. Then f has zero Taylor expansion at x, so f vanishes on an open neighbourhood of x. In particular, x lies in the interior of $V' \cap V_k$. So the interior of $V' \cap V_k$ is both open and closed, and $V' \subset V_k$ since V' is connected. □

Lemma 2.2 (Exercise 8 of [2]) *Let $\mathcal{G}$ be an algebraic group over $\mathbb{R}$ in which $\mathbb{F}$ embeds. Then for any limit group G there exists an embedding*

$$G \hookrightarrow \mathcal{G}.$$

In particular, G embeds into $SL_2(\mathbb{R})$ and $SO(3)$.

Proof. Consider the variety $V = \operatorname{Hom}(G, \mathcal{G})$. (If G is of rank r, then V is a subvariety of $\mathcal{G}^r$, cut out by the relations of G. By Hilbert's Basis Theorem, finitely many relations suffice.) For each $g \in G$, consider the subvariety

$$V_g = \{f \in V | f(g) = 1\}.$$

If G does not embed into $\mathcal{G}$ then V is covered by the subvarieties V_g for $g \neq 1$. By lemma 2.1 every component of V_{reg} is contained in some V_g, so

$$V = V_{g_1} \cup \ldots \cup V_{g_n}$$

for some non-trivial $g_1, \ldots, g_n \in G$.

So every homomorphism from G to $\mathcal{G}$ kills one of the g_i. But $\mathbb{F}$ embeds in $\mathcal{G}$, so this contradicts the assumption that G is ω-residually free. □

Remark 2.3 *Given a limit group G and an embedding $f : G \hookrightarrow SL_2(\mathbb{R})$ we have a natural map $G \to PSL_2(\mathbb{R})$. This is also an embedding since any element in its kernel satisfies $f(g)^2 = 1$ and G is torsion-free.*

We can gain more control over embeddings into $PSL_2(\mathbb{R})$ by considering the trace function.

Lemma 2.4 (Exercise 9 of [2]) *If G is a limit group and $g_1, \ldots, g_n \in G$ are non-trivial then there is an embedding $G \hookrightarrow PSL_2(\mathbb{R})$ whose image has no non-trivial parabolic elements, and so that the images of $g_1, \ldots, g_n$ are all hyperbolic.*

Proof. We abusively identify each element of the variety $V = \mathrm{Hom}(G, SL_2(\mathbb{R}))$ with the corresponding element of $\mathrm{Hom}(G, PSL_2(\mathbb{R}))$, and call it elliptic, hyperbolic or parabolic accordingly. For each $g \in G$, consider the closed subvariety U_g of homomorphisms that map g to a parabolic, and the open set W_g of homomorphisms that map g to a hyperbolic. (Note that $\gamma \in SL_2(\mathbb{R})$ is parabolic if $|\mathrm{tr}\gamma| = 2$ and hyperbolic if $|\mathrm{tr}\gamma| < 2$.) Fix an embedding $\mathbb{F} \hookrightarrow SL_2(\mathbb{R})$ whose image in $PSL_2(\mathbb{R})$ is the fundamental group of a sphere with open discs removed; such a subgroup is called a *Schottky group*, and every non-trivial element is hyperbolic. Call a component V' of V_{reg} *essential* if its closure contains a homomorphism $G \to SL_2(\mathbb{R})$ that factors through $\mathbb{F} \hookrightarrow SL_2(\mathbb{R})$ and maps the g_i non-trivially.

Suppose every essential component V' of V_{reg} is contained in some $U_{g'}$ for non-trivial g'. Then, since there are only finitely many components, for certain non-trivial $g'_1, \ldots, g'_m$, every homomorphism $G \to \mathbb{F}$ kills one of the g_i or one of the g'_j, contradicting the assumption that G is ω-residually free. Therefore, by lemma 2.1, there exists an essential component V' so that $V' \cap \bigcup_{g \neq 1} U_g$ has empty interior. In particular,

$$\bigcap_i W_{g_i} \smallsetminus \bigcup_{g \neq 1} U_g$$

is non-empty, as required. $\square$

3 GADs for limit groups

In this section we provide solutions to exercises 10 and ll, and also the related exercise 17. For the definitions of the modular group $\mathrm{Mod}(G)$, and of generalized Dehn twists, see definitions 1.6 and 1.17 respectively in [2].

Lemma 3.1 (Exercise 10 of [2]) $\mathrm{Mod}(G)$ *is generated by inner automorphisms and generalized Dehn twists.*

Proof. Since the mapping class group of a surface is generated by Dehn twists (see, for example, [4]), it only remains to show that unimodular automorphisms of abelian vertices are generated by generalized Dehn twists. Given such a vertex A, we can write

$$G = A *_{\bar{P}(A)} B$$

for some subgroup B of G. Any unimodular automorphism of A is a generalized Dehn twist of this splitting. $\square$

Lemma 3.2 (Exercise 11 of [2]) *Let M be a non-cyclic maximal abelian subgroup of a limit group G.*

1. *If $G = A *_C B$ for C abelian then M is conjugate into either A or B.*
2. *If $G = A*_C$ with C abelian then either M is conjugate into A or there is a stable letter t so that M is conjugate to $M' = \langle C, t\rangle$ and $G = A *_C M'$.*

Proof. We first prove 1. Suppose M is not elliptic in the splitting $G = A *_C B$. (Note that we don't yet know that M is finitely generated.) Non-cyclic abelian groups have no free splittings, so C is non-trivial. Let T be the Bass–Serre tree of the splitting. Either M fixes an axis in T, or it fixes a point on the boundary. In the latter case, there is an increasing chain of edge groups

$$C_1 \subset C_2 \subset \ldots M.$$

But every C_i is a conjugate of C_1, and since M is malnormal it follows that $C_i = C_1$. So $M = C_i$, contradicting the assumption that M is not elliptic.

If M fixes a line in T then M can be conjugated to M' fixing a line L so that A stabilizes a vertex v of L and M' is of the form $M' = C \oplus \mathbb{Z}$; C fixes L pointwise, and $\mathbb{Z}$ acts as translations of L. Consider the edges of L incident at v, corresponding to the cosets C and aC, for some $a \in A \smallsetminus C$. Since $aCa^{-1} = C$ and C is non-trivial, it follows from lemma 1.11 that $a \in M'$. But a is elliptic so $a \in C$, a contradiction.

In the HNN-extension case, assuming M is not elliptic in the splitting we have as before that M preserves a line in the Bass–Serre tree T. Conjugating M to M', we may assume C fixes an edge in the preserved line L, so $C \subset M'$. The stabilizer of an adjacent edge is of the form $(ta)C(ta)^{-1}$, where $a \in A$ and t is the stable letter of the HNN-extension. Therefore $C = (ta)C(ta)^{-1}$, so since G is CSA it follows that $M' = C \oplus \langle ta\rangle$ and

$$G = A *_C M'$$

as required. □

Remark 3.3 *Note that, in fact, the proof of lemma 1.11 only used that the vertex groups are CSA and that the edge group was maximal abelian on one side.*

A one-edge splitting of G is said to satisfy *condition JSJ* if every non-cyclic abelian group is elliptic in it. Recall that a limit group is *generic* if it is freely indecomposable, non-abelian and not a surface group.

Lemma 3.4 (Exercise 17 of [2]) *If G is a generic limit group then* $\mathrm{Mod}(G)$ *is generated by inner automorphisms and generalized Dehn twists in one-edge splittings satisfying JSJ.*

Furthermore, the only generalized Dehn twists that are not Dehn twists can be taken to be with respect to a splitting of the from $G = A*_C B$ with $A = C\oplus\mathbb{Z}$.

Proof. By lemma 3.1, Mod(G) is generated by inner automorphisms and generalized Dehn twists. By lemma 3.2, any splitting of G as an amalgamated product satisfies JSJ. Consider, therefore, the splitting

$$G = A *_C .$$

A (generalized) Dehn twist δ_z in this splitting fixes A and maps the stable letter $t \mapsto tz$, for some $z \in Z_G(C)$.

Suppose that this HNN-extension doesn't satisfy JSJ, so there exists some (without loss, maximal) abelian subgroup M that is not elliptic in the splitting. By lemma 3.2, after conjugating M to M', we have that

$$G = A *_C M'$$

and $M' = C \oplus \langle t \rangle$ where t is the stable letter. Since $M' = Z_G(C)$, a Dehn twist δ_z along z (for $c \in C$ and $n \in \mathbb{Z}$) fixes A and maps

$$t \mapsto z + t.$$

But this is a generalized Dehn twist in the amalgamated product. So Mod(G) is, indeed, generated by generalized Dehn twists in one-edge splittings satisfying JSJ.

Any generalized Dehn twist δ that is not a Dehn twist is in a splitting of the form

$$G = A *_C B$$

with A abelian, and acts as a unimodular automorphism on A that preserves $\bar{P}(A)$. Recall that $A/\bar{P}(A)$ is finitely generated, by remark 1.15 of [2]. To show that it is enough to use splittings in which $A = C \oplus \mathbb{Z}$, we work by induction on the rank of $A/\bar{P}(A)$. Write $A = A' \oplus \mathbb{Z}$, where $\bar{P}(A) \subset A'$. Then there is a modular automorphism α, agreeing with δ on A', generated by generalized Dehn twists of the required form, by induction. Now δ and α differ by a generalized Dehn twist in the splitting

$$G = A *_{A'} (A' *_C B)$$

which is of the required form. □

4 Constructible Limit Groups

For the definition of CLGs see [2]. The definition lends itself to the technique of proving results by a nested induction, first on level and then on the number of edges of the GAD Δ. To prove that CLGs have a certain property, this technique often reduces the proof to the cases where G has a one-edge splitting over groups for which the property can be assumed. In this section we provide solutions to exercises 12, 13, 14 and 15, which give the first properties of CLGs, culminating in the result that all CLGs are ω-residually free.

4.1 CLGs are CSA

We have seen that limit groups are CSA. This section is devoted to proving that CLGs are also CSA. Knowing this will prove extremely useful in deducing the other properties of CLGs. Note that the property of being CSA passes to subgroups.

Lemma 4.1 *CLGs are CSA.*

By induction on the number of edges in the graph of groups Δ, it suffices to consider a CLG G such that

$$G = A *_C B$$

or

$$G = A*_C$$

where each vertex group is assumed to be CSA and the edge group is taken to be maximal abelian on one side. (In the first case, we will always assume that C is maximal abelian in A.) First, we have an analogue of lemma 3.2.

Lemma 4.2 *Let G be as above. Then G decomposes as an amalgamated product or HNN-extension in such a way that all non-cyclic maximal abelian subgroups are conjugate into a vertex group. Furthermore, in the HNN-extension case we have that $C \cap C^t = 1$ where C is the edge group and t is the stable letter.*

Proof. By induction, we can assume that the vertex groups are CSA. Note that the proof of the first assertion of lemma 3.2 only relies on the facts that the vertex groups are CSA and the edge group is maximal abelian in one vertex group. So the amalgamated product case follows.

Now consider the case of an HNN-extension. Suppose that, for some stable letter t, $C \cap C^t$ is non-trivial. Then since the vertex group A is commutative transitive, it follows that $C \subset C^t$ (or $C^t \subset C$, in which case replace t by t^{-1}). If $\rho : G \to G'$ is the retraction to a lower level then, since G' can be taken to be CSA, $\rho(C^t) = \rho(C)$. But ρ is injective on edge groups so $C = C^t$ and, furthermore, t commutes with C.

Otherwise, for every choice of stable letter t, $C \cap C^t$ is trivial. In either case, the result now follows as in the proof of the second assertion of lemma 3.2. □

Recall that a simplicial G-tree is *k-acylindrical* if the fixed point set of every non-trivial element of G has diameter at most k.

Lemma 4.3 *In the graph-of-groups decomposition given by lemma 4.2, the Bass–Serre tree is 2-acylindrical.*

Proof. In the amalgamated product case this is because, for any $a \in A \smallsetminus C$, $C \cap C^a = 1$. Likewise, in the HNN-extension case this is because $C \cap C^t = 1$ where t is the stable letter. □

Proof of lemma 4.1. Let $M \subset G$ be a maximal abelian subgroup and suppose

$$1 \neq m \subset M^g \cap M.$$

Let T be the Bass–Serre tree of the splitting. If M is cyclic then it might act as translations on a line L in T. Then g also maps L to itself. But it follows from acylindricality that any element that preserves L lies in M; so $g \in M$ as required.

We can therefore assume that M fixes a vertex v of T. If g also fixes v then, since the vertex stabilizers are CSA, $g \in M$. Consider the case when

$$G = A *_C B;$$

the case of an HNN-extension is similar. Then without loss of generality $M \subset A$ and $g = ba$ for some $a \in A$ and $b \in B$ so M^g fixes a vertex stabilized by A^b for some $b \in B$. Since C is maximal abelian in A we have $C = M$ and since B is CSA and $m \in C \cap C^b$ it follows that b commutes with C so $b \in M$, hence $g \in M$. □

4.2 Abelian subgroups

It is no surprise that CLGs share the most elementary property of limit groups.

Lemma 4.4 *CLGs are torsion-free.*

Proof. The freely decomposable case is immediate by induction. Therefore assume G is a freely indecomposable CLG of level n, with Δ and $\rho : G \to G'$ as in the definition. Suppose $g \in G$ is of finite order. Then g acts elliptically on the Bass–Serre tree of Δ, so g lies in a vertex group. Clearly if the vertex is QH then $g = 1$, and by induction if the vertex is rigid then $g = 1$. Suppose therefore that the vertex is abelian. Then g lies in the peripheral subgroup. But ρ is assumed to inject on the peripheral subgroup, so by induction g is trivial. □

However, it is far from obvious that limit groups have abelian subgroups of bounded rank; indeed, it is not obvious that all abelian subgroups of limit groups are finitely generated. But this is true of CLGs.

Lemma 4.5 (Exercise 13 in [2]) *Abelian subgroups of CLGs are free, and there is a uniform (finite) bound on their rank.*

Proof. The proof starts by induction on the level of G. Let G be a CLG. Since non-cyclic abelian subgroups have no free splittings we can assume G is freely indecomposable. Let Δ be a generalized abelian decomposition. Let T be the Bass–Serre tree of Δ. If A fixes a vertex of T then the result follows by induction on level. Otherwise, A fixes a line T_A in T, on which it acts by translations. The quotient $\Delta' = T_A/A$ is topologically a circle; after some collapses Δ' is an HNN-extension; so the rank of A is bounded by the maximum rank of abelian subgroups of the vertex groups plus 1. So by induction the rank of A is uniformly bounded. That A is free follows from lemma 4.4. □

4.3 Heredity

Let Σ be a (not necessarily compact) surface with boundary. Then a boundary component δ is a circle or a line, and defines up to conjugacy a cyclic subgroup $\pi_1(\delta) \subset \pi_1(\Sigma)$. These are called the *peripheral subgroups* of Σ.

Remark 4.6 *Let Σ be a non-compact surface with non-abelian fundamental group. Then there exists a non-trivial free splitting of $\pi_1(\Sigma)$, with respect to which all peripheral subgroups are elliptic.*

Lemma 4.7 (Exercise 12 in [2]) *Let G be a CLG of level n and H a finitely generated subgroup. Then H is a free product of finitely many CLGs of level at most n.*

Proof. The subgroup H can be assumed to be freely indecomposable by Grushko's theorem, so we can also assume that G is freely indecomposable.

Let Δ and $\rho : G \to G'$ be as in the definition. Then subgroup H inherits a graph-of-groups decomposition from Δ, namely the quotient of the Bass–Serre tree T by H. Since H is finitely generated, it is the fundamental group of some finite core $\Delta' \subset T/H$. Every vertex of Δ' covers a vertex of Δ, from which it inherits its designation as QH, abelian or rigid.

The edge groups of Δ' are subgroups of the edge groups of Δ, so they are abelian and ρ is injective on them. Furthermore, it follows from lemma 4.1 that H is commutative transitive, so each edge group of Δ' is maximal abelian on one side of the associated one-edge splitting.

Let V' be a vertex group of Δ', a subgroup of the vertex group V of Δ. There are three case to consider.

1. $V' \subset V$ are abelian. Since every map $f' : V' \to \mathbb{Z}$ with $f'(P(V')) = 0$ extends to a map $f : V \to \mathbb{Z}$ with $f(P(V)) = 0$ we have that
$$\bar{P}(V') \subset \bar{P}(V)$$
so ρ is injective on $\bar{P}(V')$.

2. $V' \subset V$ are QH. If V' is of infinite index in V then V' is the fundamental group of a non-compact surface, so by remark 4.6 H is freely decomposable. Therefore it can be assumed that V' is of finite degree m in V. In particular, V is the fundamental group of a compact surface that admits a pseudo-Anosov automorphism. Furthermore, let $g, h \in V$ be such that $\rho([g, h]) \neq 1$. Then because CLGs are commutative transitive,
$$\rho([g^m, h^m]) \neq 1.$$
But $g^m, h^m \in V'$, so $\rho(V')$ is non-abelian.

3. $V' \subset V$ are rigid. Then $\tilde{V}' \subset \tilde{V}$ because CLGs are commutative transitive, so $\rho|_{V'}$ is injective.

Therefore $\rho|_H : H \to G'$ and Δ' satisfy the properties for H to be a CLG. $\square$

4.4 Coherence

A group is *coherent* if every finitely generated subgroup is finitely presented. Note that free groups and free abelian groups are coherent. For limit groups, coherence is an instance of a more general phenomenon, as in the next lemma. Recall that a group is *slender* if every subgroup is finitely generated. Finitely generated abelian groups are slender.

Lemma 4.8 *The fundamental group of a graph of groups with coherent vertex groups and slender edge groups is coherent.*

Proof. Let Δ be a graph of groups, with coherent vertex groups and slender edge groups. Let $G = \pi_1(\Delta)$ and $H \subset G$ a finitely generated subgroup. Then H inherits a graph-of-groups decomposition from Δ given by taking the quotient of the Bass–Serre tree T of Δ by the action of H. Since H is finitely generated it is the fundamental group of some finite core $\Delta' \subset T/H$. But, by induction on the number of edges, $H = \pi_1(\Delta')$ is finitely presented. □

Lemma 4.9 (Exercise 12 in [2]) *CLGs are coherent, in particular finitely presented.*

Proof. In the case of a free decomposition the result is immediate. In the other case, the (free abelian) edge groups of Δ are finitely generated, so slender and coherent, by lemma 4.5. Therefore all vertex groups are finitely generated; in particular, abelian vertex groups are coherent. Finitely generated surface groups are also coherent. Rigid vertex groups embed into a CLG of lower level, so by lemma 4.7 they are free products of coherent groups and hence coherent by induction. The result now follows by lemma 4.8. □

4.5 Finite $K(G, 1)$

That CLGs have finite $K(G, 1)$ follows from the fact that graphs of aspherical spaces are aspherical.

Theorem 4.10 (Proposition 3.6 of [9]) *Let Δ be a graph of groups; suppose that for every vertex group V there exists finite $K(V, 1)$, and for every edge group E there exists a finite $K(E, 1)$. Then for $G = \pi_1(\Delta)$, there exists a finite $K(G, 1)$.*

Surface groups and abelian groups have finite Eilenberg–Mac Lane spaces. Rigid vertices embed into a CLG of lower level, so by lemma 4.7 and induction they also have finite Eilenberg–Mac Lane spaces.

Corollary 4.11 (Exercise 13 in [2]) *If G is a CLG then there exists a finite $K(G, 1)$.*

4.6 Principal cyclic splittings

A *principal cyclic splitting* of G is a one-edge splitting of G with cyclic edge group, such that the image of the edge group is maximal abelian in one of the vertex groups; further, if it is an HNN-extension then the edge group is required to be maximal abelian in the whole group. The key observation about principal cyclic splittings is that any non-cyclic abelian subgroup is elliptic with respect to them—in other words, they are precisely those cyclic splittings that feature in the conclusion of lemma 4.2. Applying lemma 4.2, to prove that every freely indecomposable, non-abelian CLG has a principal cyclic splitting it will therefore suffice to produce any non-trivial cyclic splitting (since we now know that CLGs are CSA).

Proposition 4.12 (Exercise 14 in [2]) *Every non-abelian, freely indecomposable CLG admits a principal cyclic splitting.*

Proof. Let G be a CLG. As usual, by induction it suffices to consider the cases when G splits as an amalgamated product or HNN-extension. It suffices to exhibit any cyclic splitting of G, as observed above.

Suppose

$$G = A *_C B.$$

If C is cyclic the result is immediate, so assume C is non-cyclic abelian. If either vertex group is freely decomposable then so is G, since C has no free splittings; if both vertex groups are abelian then so is G. Therefore A, say, is freely indecomposable and non-abelian so has a principal cyclic splitting, which we shall take to be of the form

$$A = A' *_{C'} B'.$$

(It might also be an HNN-extension, but this doesn't affect the proof.) Because it is principal C is conjugate into a vertex, say B'; so G now decomposes as

$$G = A' *_{C'} (B' *_C B).$$

which is a cyclic splitting as required.

The proof when $G = A *_C$ is the same. □

4.7 A criterion in free groups

To prove that a group G is ω-residually free, it suffices to show that for any finite $X \subset G \smallsetminus 1$ there exists a homomorphism $f : G \to \mathbb{F}$ with $1 \notin f(X)$. So a criterion to show that an element of $\mathbb{F}$ is not the identity will be useful.

Lemma 4.13 *Let $z \in \mathbb{F} \smallsetminus 1$, and consider an element g of the form*

$$g = a_0 z^{i_1} a_1 z^{i_2} a_2 \dots a_{n-1} z^{i_n} a_n$$

where $n \geq 1$ and, whenever $0 < k < n$, $[a_k, z] \neq 1$. Then $g \neq 1$ whenever the $|i_k|$ are sufficiently large.

Choose a generating set for $\mathbb{F}$ so the corresponding Cayley graph is a tree T. An element $u \in \mathbb{F}$ specifies a geodesic $[1,u] \subset T$. Likewise, a string of elements $u_0, u_1, \ldots, u_n \in \mathbb{F}$ defines a path

$$[1,u_0] \cdot u_0[1,u_1] \cdot \ldots \cdot (u_0 \ldots u_{n-1})[1,u_n]$$

in T, where $\cdot$ denotes concatenation of paths. The key observation we will use is as follows. The length of a word $w \in \mathbb{F}$ is denoted by $|w|$.

Remark 4.14 *Suppose z is cyclically reduced and has no proper roots. Let $a \in \mathbb{F}$ be such that a and az both lie in $L \subset T$ the axis of z. If j is minimal such that z^j lies in the geodesic $[a, az]$ then, setting $u = z^j a^{-1}$ and $v = az^{1-j}$, it follows that $uv = z = vu$; in particular, either u or v is trivial and $[a,z] = 1$.*

Proof of lemma 4.13. It can be assumed that z is cyclically reduced and has no proper roots.

Assume that, for each k, $|z^{i_k}| \geq |a_{k-1}| + |a_k| + |z|$. Let $L \subset T$ be the axis of z. Denote by g_k the partial product

$$g_k = a_0 z^{i_1} a_1 z^{i_2} a_2 \ldots a_{k-1} z^{i_k} a_k.$$

The path γ corresponding to g is of the form

$$[1,a_0] \cdot g_0[1,z^{i_1}] \cdot g_0 z^{i_1}[1,a_1] \cdot \ldots \cdot g_{n-1}[1,z^{i_n}] \cdot g_{n-1} z^{i_n}[1,a_n].$$

Suppose that $g = 1$ so this path is a loop. Each section of the form $g_k[1,z^{i_k}]$ lies in a translate of L, the axis of z. Since T is a tree, for at least one such section γ enters and leaves $g_k L$ at the same point—otherwise γ is a non-trivial loop. Since $|z^{i_k}| > |a_{k-1}| + |a_k| + |z|$ it follows that both $g_k z^{i_k} a_k$ and $g_k z^{i_k} a_k z$ lie in $g_k L$ and so $[a_k, z] = 1$ by remark 4.14. $\square$

4.8 CLGs are limit groups

Theorem 4.15 (Exercise 15 in [2]) *CLGs are ω-residually free.*

Since the freely decomposable case is immediate, let Δ, G' and ρ be as in the definition of a CLG in [2]. By induction, G' can be assumed ω-residually free. As a warm up, and for use in the subsequent induction, we first prove the result in the case of abelian and surface vertices.

Lemma 4.16 *Let A be a free abelian group and $\rho : A \to G'$ a homomorphism to a limit group. Suppose $P \subset A$ is a nontrivial subgroup of finite corank closed under taking roots, on which ρ is injective. Then for any finite subset $X \subset A \smallsetminus 1$ there exists an automorphism α of A, fixing P, so that $1 \notin \rho \circ \alpha(X)$.*

Proof. Since $\ker \rho$ is a subgroup of A of positive codimension, for given $x \in A \smallsetminus 0$ a generic automorphism α certainly satisfies $\alpha(x) \notin \ker \rho$. Since X is finite, therefore, there exists α such that $\alpha(x) \notin \ker \rho$ for any $x \in X$. $\square$

We now consider the surface vertex case.

Proposition 4.17 *Let S be the fundamental group of a surface Σ with non-empty boundary, with $\chi(\Sigma) \leq -1$, and $\rho : S \to G'$ a homomorphism injective on each peripheral subgroup and with non-abelian image. Then for any finite subset $X \subset S \setminus 1$ there exists an automorphism α of S, induced by an automorphism of Σ fixing the boundary components pointwise, such that $1 \notin \rho \circ \alpha(X)$.*

Let the surface Σ have $b > 0$ boundary components and Euler characteristic $\chi < 0$. When Σ is cut along a two-sided simple closed curve γ, the resulting pieces either have lower genus (defined to be $1 - \frac{1}{2}(\chi + b)$) or fundamental groups of strictly lower rank, depending on whether γ was separating or not. The simplest cases all have fundamental groups that are free of rank 2.

Example 4.18 (The simplest cases) *Suppose S is free of rank 2. By lemma 1.7, $\rho(S)$ is free or free abelian; but $\rho(S)$ is assumed non-abelian, so $\rho(S)$ is free and ρ is injective.*

For the more complicated cases, the idea is to find a suitable simple closed curve ζ along which to cut to make the surface simpler. In order to apply the proposition inductively, ζ needs the following properties:

1. $\rho(\zeta) \neq 1$;
2. the fundamental group S' of any component of $\Sigma \setminus \zeta$ must have $\rho(S')$ non-abelian.

Let's find this curve in some examples.

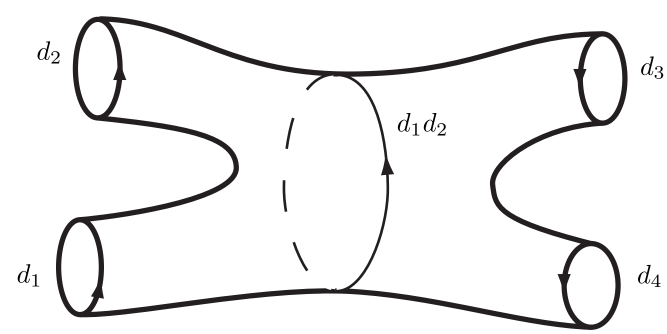

Figure 1: A four-times punctured sphere

Example 4.19 (Punctured spheres) *Suppose Σ is a punctured sphere, so*

$$S = \langle d_1, \ldots, d_n | \prod_j d_j \rangle.$$

Assume $n \geq 4$ and $\rho(d_i) \neq 1$ for all i. Define a relation on $\{1, \ldots, n\}$ by

$$i \sim j \Leftrightarrow \rho([d_i, d_j]) = 1.$$

Since G' is commutative transitive, $\sim$ is an equivalence relation. Because the image is non-abelian, there are at least two equivalence classes. Since

$$\prod_i d_i = 1$$

any equivalence class has at least two elements in its complement. Relabelling if necessary, it can now be assumed that

$$\rho([d_1, d_2]), \rho([d_3, d_4]) \neq 1.$$

Now if the boundary curves have been coherently oriented then $d_1 d_2$ has a representative that is a simple closed curve. Take ζ as this representative.

The case when Σ is non-orientable is closely related.

Example 4.20 (Non-orientable surfaces) *Suppose Σ is non-orientable so*

$$S = \langle c_1, \ldots, c_m, d_1, \ldots, d_n | \prod_i c_i^2 \prod_j d_j \rangle.$$

Exactly the same argument as in the case of a punctured sphere would work if it could be guaranteed that $\rho(c_i) \neq 1$ for all i.

Fix some c_k, therefore, and suppose $\rho(c_k) = 1$. Let γ be a simple closed curve representing c_k. Then $d_1 c_k$ has a representative δ which is a simple closed curve, and $\rho(d_1 c_k) \neq 1$. Furthermore, $\Sigma \smallsetminus \gamma$ and $\Sigma \smallsetminus \delta$ are homeomorphic surfaces, and a homeomorphism between them extends to an automorphism of Σ mapping γ to δ. This homeomorphism can be chosen not to alter any of the other c_i or the d_j.

Therefore, after an automorphism of Σ, it can be assumed that $\rho(c_i) \neq 1$ for all i, so a suitable ζ can be found as in the previous example.

Example 4.21 (Positive-genus surfaces) *Suppose Σ is an orientable surface of positive genus, so*

$$S = \langle a_1, b_1, \ldots, a_g, b_g, d_1, \ldots, d_n | \prod_i [a_i, b_i] \prod_j d_j \rangle.$$

Assume that $g, n \geq 1$. If, for example, $\rho(a_1) \neq 1$ then ζ can be taken to be a simple closed curve representing a_1. Otherwise, $\rho(a_1 d_1) \neq 1$ and $a_1 d_1$ has a simple closed representative. It remains to show that the single component of $\Sigma \smallsetminus \zeta$ has non-abelian image.

Cutting along ζ expresses S as an HNN-extension:

$$S = S' *_{\mathbb{Z}} .$$

Let t be the stable letter, and suppose $\rho(S')$ is abelian. Then $\zeta \in tS't^{-1} \cap S'$ so in particular $\rho(tS't^{-1} \cap S')$ is non-trivial. But G' is a limit group and hence CSA, so $\rho(t)$ commutes with $\rho(S')$, contradicting the assumption that $\rho(S)$ is non-abelian.

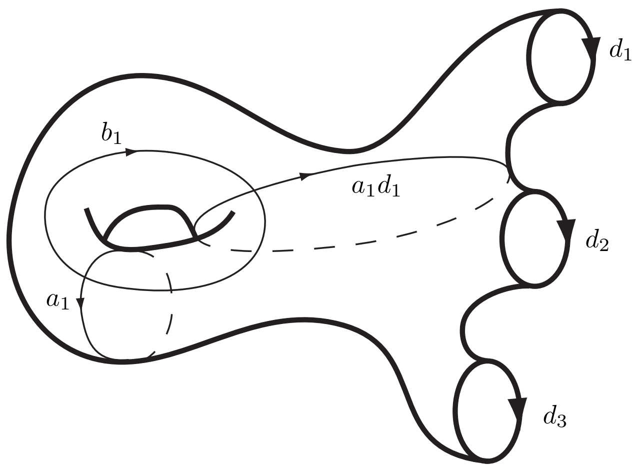

Figure 2: The positive-genus case.

Note that examples 4.19, 4.20 and 4.21 cover all the more complicated surfaces with boundary.

Proof of proposition 4.17. Example 4.18 covers all the simplest cases. Suppose therefore that Σ is more complicated. To apply the inductive hypothesis, an essential simple closed curve $\zeta \in \Sigma$ is needed such that $\rho(\zeta) \neq 1$ and, for any component S' of $\Sigma \smallsetminus \zeta$, $\rho(S')$ is non-abelian. This is provided by examples 4.19, 4.20 and 4.21.

For simplicity, assume ζ is separating. The non-separating case is similar. Then $\Sigma \smallsetminus \zeta$ has two components, Σ_1 and Σ_2. Let $S_i = \pi_1(\Sigma_i)$, and denote by X_i the *syllables* of X in S_i—the elements of S_i that occur in the normal form of some $x \in X$ with respect to the splitting over $\langle\zeta\rangle$. Because the pieces Σ_i are simpler than Σ there exists $\alpha \in \mathrm{Aut}_0(\Sigma)$ and $f : G' \to \mathbb{F}$ such that

$$1 \notin f \circ \rho \circ \alpha([\zeta, X_1 \cup X_2]).$$

Consider $\xi \in X$. The proposition follows from the claim that, for all sufficiently large k,

$$f \circ \rho \circ \delta_\zeta^k \circ \alpha(\xi) \neq 1$$

where δ_ζ is a Dehn twist in ζ. If ξ is a power of ζ then the result is immediate. Otherwise, with respect to the one-edge splitting of G over $\langle\zeta\rangle$, ξ has reduced form

$$\sigma_0\tau_0\sigma_1\tau_1 \ldots \sigma_n\tau_n$$

where the $\sigma_i \in X_1$ and the $\tau_j \in X_2$. The image $x^{(k)} = f \circ \rho \circ \delta_\zeta^k \circ \alpha(\xi)$ is of the form

$$z^k s_0 z^{-k} t_0 z^k s_1 z^{-k} t_1 \ldots z^k s_n z^{-k} t_n$$

where $z = f \circ \rho(\zeta)$, $s_i = f \circ \rho \circ \alpha(\sigma_i)$ and $t_i = f \circ \rho \circ \alpha(\tau_i)$. This expression for $x^{(k)}$ satisfies the hypotheses of lemma 4.13, so $x^{(k)} \neq 1$ for all sufficiently large k. □

The proof of theorem 4.15 is very similar to the proof of proposition 4.17. The theorem follows from the following proposition, by induction on level.

Proposition 4.22 *Let G be a freely indecomposable CLG, let G' be ω-residually free, and let Δ and ρ be as usual. For any finite subset $X \subset G \smallsetminus 1$ there exists a modular automorphism α of G such that $1 \notin \rho \circ \alpha(X)$.*

Proof. As usual, the proposition is proved by induction on the number of edges of Δ. The case of Δ having no edges follows from lemmas 4.16 and 4.17, and the fact that ρ is injective on rigid vertices. By induction on level, G' is a limit group.

Now suppose Δ has an edge group E. For simplicity, assume E is separating. The non-separating case is similar. Then removing the edge corresponding to E divides Δ into two subgraphs Δ_1 and Δ_2. Let $G_i = \pi_1(\Delta_i)$, and denote by X_i the syllables of X in G_i. Without loss assume E is maximal abelian in G_1. Fix non-trivial $\zeta \in E$. By induction there exists $\alpha \in \mathrm{Mod}(\Delta)$ and $f : G' \to \mathbb{F}$ such that

$$1 \notin f \circ \rho \circ \alpha([\zeta, X_1] \cup X_2).$$

Consider $\xi \in X$. The proposition follows from the claim that, for all sufficiently large k,

$$f \circ \rho \circ \delta_\zeta^k \circ \alpha(\xi) \neq 1.$$

If $\xi \in E$ then the result is immediate. Otherwise, with respect to the one-edge splitting of G over E, ξ has reduced form

$$\sigma_0 \tau_0 \sigma_1 \tau_1 \ldots \sigma_n \tau_n$$

where the $\sigma_i \in X_1$ and the $\tau_j \in X_2$. The image $x^{(k)} = f \circ \rho \circ \delta_\zeta^k \circ \alpha(\xi)$ is of the form

$$z^k s_0 z^{-k} t_0 z^k s_1 z^{-k} t_1 \ldots z^k s_n z^{-k} t_n$$

where $z = f \circ \rho(\zeta)$, $s_i = f \circ \rho \circ \alpha(\sigma_i)$ and $t_i = f \circ \rho \circ \alpha(\tau_i)$. In particular, canceling across those t_i that commute with z, we have

$$x^{(k)} = u_0 z^{k\epsilon_1} u_1 \ldots u_{n-1} z^{k\epsilon_n} u_n$$

where $\epsilon_i = \pm 1$ and u_i don't commute with z for $0 < i < n$. This second expression for $x^{(k)}$ satisfies the hypotheses of lemma 4.13, so $x^{(k)} \neq 1$ for all sufficiently large k. □

5 The Shortening Argument

We consider a sequence of G-trees T_i, arising from homomorphisms $f_i : G \to \mathbb{F}$, that converge in the Gromov topology to a G-tree T. By the results of section 3 of [2], if the action of G on the limit tree T is faithful then it gives rise to a generalized abelian decomposition for G. This section is entirely devoted to the solution of exercise 16, which is essentially Rips and Sela's shortening argument—an ingenious means of using this generalized abelian decomposition to force the action on the limit tree to be unfaithful.

5.1 Preliminary ideas

Once again, fix a generating set for $\mathbb{F}$ so that the corresponding Cayley graph is a tree, and let $|w|$ denote the length of a word $w \in \mathbb{F}$. Fix a generating set S for G. For $f : G \to \mathbb{F}$, let

$$|f| = \max_{g \in S} |f(g)|.$$

A homomorphism is *short* if

$$|f| \leq |\iota \circ f \circ \alpha|$$

whenever α is a modular automorphism of G and ι is an inner automorphism of $\mathbb{F}$.

Theorem 5.1 (Exercise 16 of [2]) *Suppose every f_i is short. Then the action on T is not faithful.*

The proof is by contradiction. We assume therefore, for the rest of section 5, that the action is faithful. By the results summarized in section 3 of [2], the action of G on T gives a GAD Δ for G. The idea is, if T_i are the limiting trees with basepoints x_i, to construct modular automorphisms ϕ_n so that

$$d_i(x_i, f_i \circ \phi_i(g)x_i) < d_i(x_i, f_i(g)x_i)$$

for all sufficiently large i. Then apply these automorphisms to carefully chosen basepoints.

All constructions of the limit tree T, such as the asymptotic cone [11], use some form of based convergence: basepoints $x_i \in T_i$ are fixed, and converge to a basepoint $[x_i] \in T$. Because the f_i are short,

$$\max_{g \in S} d_{\mathbb{F}}(1, f_i(g)) \leq \max_{g \in S} d_{\mathbb{F}}(t, f_i(g)t)$$

for all $t \in T_{\mathbb{F}}$; otherwise, conjugation by the element of $\mathbb{F}$ nearest to t leads to a shorter equivalent homomorphism. It follows that $1 \in T_i$ is always a valid basepoint; we set $x = [1] \in T$ to be the basepoint for T.

The proof of theorem 5.1 goes on a case-by-case basis, depending on whether $[x, gx]$ intersects a simplicial part or a minimal part of T.

5.2 The abelian part

The next proposition is a prototypical shortening result for a minimal vertex.

Proposition 5.2 *Let V be an abelian vertex group of Δ. For $g \in G$, let $l(g)$ be the translation length of g on T. Fix $\epsilon > 0$. Then for any finite subset $S \subset V$ there exists a modular automorphism ϕ of G such that*

$$\max_{g \in S} l(\phi(g)) < \epsilon.$$

Proof. The minimal V-invariant subtree T_V is a line in T, on which V acts indiscretely. Since S is finite, V can be assumed finitely generated. It suffices to prove the theorem in the case where S is a basis for V. Assume furthermore that each element of S translates T_V in the same direction.

Suppose the action of V on T_V is free. Let $S = \{g_1, \dots, g_n\}$, ordered so that

$$l(g_1) > l(g_2) > \ldots > l(g_n) > 0.$$

Since the action is indiscrete, there exists an integer λ such that

$$l(g_1) - \lambda l(g_2) < \frac{1}{2} l(g_2).$$

Applying the automorphism that maps $g_1 \mapsto g_1 - \lambda g_2$ and proceeding inductively, we can make $l(g_1)$ as short as we like.

If the action of V is not free then $V = V' \oplus V_0$ where V' acts freely on T_V and V_0 fixes T_V pointwise. Applying the free case to V' gives the result. $\square$

The aim is to prove the following theorem.

Theorem 5.3 *Let V be an abelian vertex. Then for any finite subset $S \subset G$ there exists a modular automorphism ϕ such that for any $g \in S$:*

1. *if $[x, gx]$ intersects a translate of T_V in a segment of positive length then*

$$d(x, \phi(g)x) < d(x, gx);$$

2. *otherwise, $\phi(g) = g$.*

Proof. By a result of J. Morgan (claim 3.3 of [6]—the article is phrased in terms of laminations), the path $[x, gx]$ intersects finitely many translates of T_V in non-trivial segments. Let ϵ be the minimal length of all such segments across all $g \in S$. Assume that $g \in S$ is such that $[x, gx]$ intersects a translate of T_V non-trivially.

Suppose first that x lies in a translate of T_V, so without loss of generality $x \in T_V$. Then g has a non-trivial decomposition in the GAD provided by corollary 3.16 of [2] of the form

$$g = a_0 b_1 a_1 \ldots a_n$$

where the a_i lie in V and the b_i are products of elements of other vertices and loop elements. Write $g_i = a_0 b_1 \dots b_{i-1} a_{i-1}$. The decomposition can be chosen so that each component of the geodesic $[x, gx]$ that lies in $g_i T_V$ is non-trivial. For each i, decompose $[x, b_i x]$ as

$$[x, s_i] \cdot [s_i, t_i] \cdot [t_i, b_i x]$$

where $[x, s_i]$ and $[t_i, b_i x]$ are maximal segments in T_V and $b_i T_V$ respectively. Then

$$[x, gx] = [x, a_0 s_1] \cdot g_1 [s_1, t_1] \cdot g_1 [t_1, b_1 a_1 s_2] \cdot \ldots \cdot g_n [s_n, t_n] \cdot g_n [t_n, b_n a_n x]$$

where each $[s_i, t_i]$ and $[t_i, b_i a_i s_{i+1}]$ is a non-trivial segment. Therefore

$$\begin{aligned} d(x, gx) &= d(x, a_0 s_1) + \sum_{i=1}^{n} d(s_i, t_i) + \sum_{i=1}^{n-1} d(t_i, b_i a_i s_{i+1}) + d(t_n, b_n a_n x) \\ &\geq \sum_{i=1}^{n} d(s_i, t_i) + (n+1)\epsilon. \end{aligned}$$

Since V acts indiscretely on the line T_V, by modifying the b_i by elements of V it can be assumed that

$$d(x, s_i), d(t_i, b_i x) < \frac{1}{4}\epsilon.$$

By proposition 5.2 there exists $\phi \in \mathrm{Mod}(G)$ such that $\phi(b_i) = b_i$ for all b_i and

$$d(x, \phi(a_i)x) < \frac{1}{2}\epsilon$$

for all a_i. Now as before $[x, \phi(g)x]$ decomposes as

$$[x, \phi(a_0)s_1] \cdot \phi(g_1)[s_1, t_1] \cdot \ldots \cdot \phi(g_n)[t_n, b_n \phi(a_n)x]$$

so

$$\begin{aligned} d(x, \phi(g)x) &= d(x, \phi(a_0)s_1) + \sum_{i=1}^{n} d(s_i, t_i) + \sum_{i=1}^{n-1} d(t_i, b_i \phi(a_i) s_{i+1}) \\ &\quad + d(t_n, b_n \phi(a_n)x) \\ &< d(x, \phi(a_0)x) + \sum_{i=1}^{n} d(s_i, t_i) + \sum_{i=1}^{n-1} d(b_i x, b_i \phi(a_i)x) \\ &\quad + d(b_n x, b_n \phi(a_n)x) + \frac{n}{2}\epsilon \\ &< \sum_{i=1}^{n} d(s_i, t_i) + \big(n + \frac{1}{2}\big)\epsilon. \end{aligned}$$

Therefore, $d(x, \phi(g)x) < d(x, gx) - \frac{1}{2}\epsilon$ and in particular the result follows.

Now suppose x does not lie in a translate of T_V. Then g has a non-trivial decomposition in the corresponding GAD of the form

$$g = b_0 a_1 b_1 \ldots b_n$$

where the a_i lie in V and the b_i are products of elements of other vertices and loop elements. Let $g' = a_1 b_1 \ldots b_{n-1} a_n$. Let x' be the first point on $[x, gx]$ in a translate of T_V, so $x' = b_0 y \in b_0 T_V$ for some $y \in T_V$. Likewise let x'' be the last point on $[x, gx]$, so $x'' = b_0 g' z \in b_0 g' T_V$ for some $z \in T_V$. Since the action of V on T_V is indiscrete we can modify b_n by an element of V and assume that $d(y, z) < \frac{1}{4}\epsilon$.

Then the geodesic $[x, gx]$ decomposes as

$$[x, gx] = [x, b_0 y] \cdot [b_0 y, b_0 g' z] \cdot [b_0 g' z, gx]$$

so

$$d(x, gx) > d(x, b_0 y) + d(y, g' y) + d(z, b_n x) - \frac{1}{4}\epsilon.$$

Applying the first case to g' and y we obtain $\phi \in \mathrm{Mod}(G)$ such that

$$d(y, \phi(g') y) < d(y, g' y) - \frac{1}{2}\epsilon$$

so

$$\begin{aligned}
d(x, \phi(g) x) &< d(x, b_0 y) + d(y, \phi(g') y) + d(z, b_n x) + \frac{1}{4}\epsilon \\
&< d(x, b_0 y) + d(y, g' y) - \frac{1}{2}\epsilon + d(z, b_n x) + \frac{1}{4}\epsilon \\
&< d(x, gx)
\end{aligned}$$

as required. □

5.3 The surface part

The surface part is dealt with by Rips and Sela, in [8], in the following theorem.

Theorem 5.4 (Theorem 5.1 of [8]) *Let V be a surface vertex. Then for any finite subset $S \subset G$ there exists a modular automorphism ϕ such that for any $g \in S$:*

1. *if $[x, gx]$ intersects a translate T_V in a segment of positive length then*

$$d(x, \phi(g) x) < d(x, gx);$$

2. *otherwise, $\phi(g) = g$.*

Rips and Sela use the notion of groups of interval exchange transformations, which are equivalent to surface groups, and prove an analogous result to proposition 5.2. The rest of the proof is the same as that of theorem 5.3.

5.4 The simplicial part

It remains to consider the case where $[x, gx]$ is contained in the simplicial part of T.

Theorem 5.5 *Let $S \subset G$ be finite and let $x \in T$. Then there exist $\phi_n \in$ $\mathrm{Mod}(\Delta)$ such that, for all $g \in S$,*

$$d(x, \phi_n(g)x) = d(x, gx);$$

furthermore, for all $g \in S$ that do not fix x, and for all sufficiently large n,

$$d_n(x_n, f_n \circ \phi_n(g)x_n) < d_n(x_n, f_n(g)x_n).$$

Let e be a closed simplicial edge containing x. The proof of the theorem is divided into cases, depending on whether the image of e is separating in T/G. In both cases, the following lemma will prove useful.

Lemma 5.6 *Let A be a vertex group of the splitting over e. Let T_A be the minimal A-invariant subtree of T; conjugating A, we can assume that $T_A \cap e$ is precisely one point, y. Fix any non-trivial $c \in C = \mathrm{Stab}(e)$. Then there exists a sequence of integers m_n such that, for any $a \in A$,*

$$d_n(x_n, f_n(c^{-m_n} a c^{m_n})x_n) \to d(y, ay)$$

as $n \to \infty$.

Proof. The key observation is that

$$2d_n(x_n, \mathrm{Axis}(f_n(c))) < d_n(x_n, f_n(c)x_n) \to 0,$$

and the same holds for the y_n. Let x'_n be the nearest point on $\mathrm{Axis}(f_n(c))$ to x_n; likewise, let y'_n be the nearest point on $\mathrm{Axis}(f_n(c))$ to y_n. Then, for each n, there exists m_n such that

$$d_n(f_n(c^{m_n})x'_n, y'_n) < l(f_n(c)) \to 0$$

as $n \to \infty$. Therefore

$$d_n(f_n(c^{m_n})x_n, y_n) \to 0$$

as $n \to \infty$, and the result follows. □

The next lemma helps with the case where the image of e is separating.

Lemma 5.7 *Assume the image of e is separating, so the induced splitting is*

$$G = A *_C B.$$

Assume furthermore that, with the notation of the previous lemma, $x \neq y$. Then there exists $\alpha_n \in \mathrm{Mod}(\Delta)$ such that, for all $g \in S$:

1. *if* $g \in A$ *then* $\alpha_n(g) = g$;
2. *if* $g \notin A$ *then*

$$d_n(x_n, f_n \circ \alpha_n(g)x_n) < d_n(x_n, f_n(g)x_n).$$

Proof. Fix a non-trivial $c \in C$, and let δ_c be the Dehn twist in c that is the identity when restricted to A. Let $\alpha_n = \delta_c^{m_n}$ where m_n are the integers given by lemma 5.6. Any $g \notin A$ has normal form

$$g = a_0 b_1 a_1 \dots b_l a_l$$

with the $a_i \in A \smallsetminus C$ and the $b_i \in B \smallsetminus C$, except for a_0 and a_l which may be trivial. Therefore $d(x, gx) = \sum_i d(x, b_i x) + \sum_i d(x, a_i x)$. Fix $\epsilon > 0$. If a_i is non-trivial then $a_i \notin C$ and so

$$d(x, a_i x) = 2d(x, y) + d(y, a_i y).$$

Let k be the number of a_i that are non-trivial (so $l - 1 \leq k \leq l + 1$). Therefore, for all sufficiently large n,

$$d_n(x_n, f_n(g)x_n) > \sum_i d(x, b_i x) + \sum_i d(y, a_i y) + 2kd(x, y) - \epsilon.$$

By contrast, for all sufficiently large n,

$$d_n(x_n, f_n \circ \alpha_n(g)x_n) < \sum_i d(x, b_i x) + \sum_i d(y, a_i y) + \epsilon$$

by lemma 5.6. By assumption $x \neq y$, so $d(x, y) > 0$. Therefore taking $\epsilon < kd(x, y)$ gives the result. $\square$

We now turn to the non-separating case.

Lemma 5.8 *Assume the image of e is non-separating, so the splitting induced by e is*

$$G = A *_C .$$

Let t be a stable letter. As before, conjugate A so that $T_A \cap e$ is precisely one point y. Fix any non-trivial $c \in C = \mathrm{Stab}(e)$. Then there exists a sequence of integers p_n such that

$$d_n(y_n, f_n(tc^{p_n})y_n) \to 0$$

as $n \to \infty$. Therefore, for any fixed integer j,

$$d_n(y_n, f_n(tc^{p_n})^j y_n) \to 0$$

as $n \to \infty$.

Proof. As in the proof of lemma 5.6, by the definition of Gromov convergence,

$$2d_n(y_n, \mathrm{Axis}(f_n(c))) < d_n(y_n, f_n(c)y_n) \to 0$$

as $n \to \infty$, and similarly,

$$2d_n(f_n(t^{-1})y_n, \mathrm{Axis}(f_n(c))) \to 0$$

as $n \to \infty$. Let y'_n be the nearest point on $\mathrm{Axis}(f_n(c))$ to y_n, and let y''_n be the nearest point on $\mathrm{Axis}(f_n(c))$ to $f_n(t^{-1})y_n$. Then there exist integers p_n such that

$$d_n(f_n(c^{p_n})y'_n, y''_n) \to 0$$

as $n \to \infty$. The result now follows. $\square$

Lemma 5.9 *Assume the situation is in lemma 5.8. Then there exists* $\alpha_n \in \mathrm{Mod}(\Delta)$ *such that, for all* $g \in S$*:*

1. *if* $g \in C$ *then* $\alpha_n(g) = g$*;*
2. *if* $g \notin C$ *then*

$$d_n(x_n, f_n \circ \alpha_n(g)x_n) < d_n(x_n, f_n(g)x_n).$$

Proof. Fix a stable letter t that translates x away from y. Fix a non-trivial $c \in C$ and let $i_c \in \mathrm{Mod}(G)$ be conjugation by c. Set $\alpha_n = i_c^{m_n} \circ \delta_c^{p_n}$, where m_n are integers given by lemma 5.6 and p_n are given by lemma 5.8. Any g is of the form

$$g = a_0 t^{j_1} a_1 \dots t^{j_l} a_l$$

with $j_i \neq 0$ and the $a_i \in A \setminus C$ except for a_0 and a_l which may be trivial. Unlike in the case of a separating edge, we have to be a little more careful in estimating $d(x, gx)$ because the natural path from x to gx given by the decomposition of g may backtrack. To be precise, backtracking occurs when $a_i \neq 1$ and $j_{i+1} < 0$ and also when $j_i > 0$ and $a_{i+1} \neq 1$. Let k be the number of i for which backtracking does not occur, so $0 \leq k \leq 2$. Then

$$d(x, gx) = \sum_i d(y, a_i y) + \sum_i d(x, t^{j_i} x) + 2kd(x, y).$$

Fix $\epsilon > 0$. Then for all sufficiently large n,

$$d_n(x_n, f_n(g)x_n) > \sum_i d(y, a_i y) + 2kd(x, y) + \sum_i d(x, t^{j_i} x) - \epsilon.$$

Now for each i,

$$d_n(f_n(c^{m_n})x_n, f_n((tc^{p_n})^{j_i})f_n(c^{m_n})x_n) \to 0$$

and

$$d_n(f_n(c^{m_n})x_n, f_n(a_i c^{m_n})x_n) \to d(y, a_i y).$$

So for all sufficiently large n,

$$d_n(x_n, f_n \circ \alpha_n(g)x_n) < \sum_i d(y, a_i y) + \epsilon.$$

Taking $2\epsilon < 2kd(x,y) + \sum_i d(x, t^{j_i}x)$ gives the result. □

We are now ready to prove the theorem.

Proof of theorem 5.5. Suppose first that x lies in the interior of an edge e. If e has separating image in the quotient then lemma 5.7 can be applied both ways round, giving rise to modular automorphisms α_n and β_n. The theorem is then proved by taking $\phi_n = \alpha_n \circ \beta_n$. If e is non-separating then applying lemma 5.9 and taking $\phi_n = \alpha_n$ gives the result.

Suppose now that x is a vertex. For each orbit of edges $[e]$ adjoining x, let α_n^e be the result of applying lemma 5.7 or lemma 5.9 as appropriate to e. Now taking

$$\phi_n = \alpha_n^{e_1} \circ \ldots \circ \alpha_n^{e_p}$$

where $[e_1], \ldots, [e_p]$ are the orbits adjoining x gives the required automorphism. □

This is the final piece of the shortening argument.

Proof of theorem 5.1. Fix a generating set S for G. Let $f_i : G \to \mathbb{F}$ be a sequence of short homomorphisms corresponding to the convergent sequence of G-trees T_i. Let T be the limiting G-tree and suppose that the action of G on T is faithful. By corollary 3.16 of [2] this induces a GAD Δ for G. Let $x \in T$ be the basepoint fixed in subsection 5.1.

Composing the automorphisms given by theorems 5.3 and 5.4 there exists $\alpha \in \mathrm{Mod}(\Delta)$ such that, for any $g \in G$,

$$d(x, \phi(g)x) < d(x, gx)$$

if $[x, gx]$ intersects an abelian or surface component of T and $\phi(g) = g$ otherwise. By theorem 5.5, for all sufficiently large i there exist $\beta_i \in \mathrm{Mod}(\Delta)$ such that $d(x, \beta_i(g)x) = d(x, gx)$ and, furthermore,

$$d_i(1, f_i \circ \beta_i(g)) < d_i(1, f_i(g))$$

whenever $[x, gx]$ is a non-trivial arc in the simplicial part of the tree. It follows that for $\phi_i = \beta_i \circ \alpha$,

$$d_i(1, f_i \circ \phi_i(g)) < d_i(1, f_i(g))$$

for all $g \in S$ and all sufficiently large i. This contradicts the assumption that the f_i were short. □

6 Bestvina and Feighn's geometric approach

In section 7 of [2], Bestvina and Feighn provide a more geometric proof of their Main Proposition. In this section we provide proofs of the exercises needed in this argument.

6.1 The space of laminations

Recall that $\mathcal{ML}(K)$ is the space of measured laminations on K, and $\mathbb{PML}(K)$ is its quotient by the action of $\mathbb{R}_+$. Let E be the set of edges of K.

Proposition 6.1 (Exercise 18 of [2]) *The space of measured laminations on K can be identified with a closed cone in $\mathbb{R}^E_+ - \{0\}$, given by the triangle inequality for each 2-cell of K. Hence, when $\mathcal{ML}(K)$ is endowed with the corresponding topology, $\mathbb{PML}(K)$ is compact.*

Proof. Recall that two laminations are considered equivalent if they assign the same measure to each edge. Therefore it suffices to show existence of a lamination with the prescribed values on the edges. First, for each edge e with $\int_e \mu > 0$, fix a closed proper subinterval I_e contained in the interior of e. Now fix a Cantor function $c_e : I_e \to [0, \int_e \mu]$. This gives a measure μ on e, given by

$$\int_J \mu = \int_{I_e \cap J} c_e d\lambda$$

where $d\lambda$ is Lebesgue measure on $\mathbb{R}$. Now suppose e_1, e_2, e_3 are the edges of a simplex in K. Divide e_1 into intervals e_1^2 and e_1^3 so that

$$2\int_{e_1^2} d\mu = \int_{e_1} d\mu + \int_{e_2} d\mu - \int_{e_3} d\mu$$

and e_1^2 shares a vertex with e_2, and similarly for e_1^3. Divide e_2 and e_3 likewise. Fix a Cantor set in each e_i^j. Now for each distinct i, j inscribe a lamination between e_i^j and e_j^i. Since any path transverse to this lamination can be homotoped to an edge path respecting the lamination, the measure on the edges determines a transverse measure to the lamination. $\square$

6.2 Matching resolutions in the limit

A measured lamination on K defines a G-tree. The next exercise shows the close relation between the topology on the space of laminations and the topology on the space of trees. For the definition of a resolution, see [2]. The solution is most easily phrased in terms of some explicit construction of the limiting tree. I shall use the asymptotic cone, T_ω; T can be realized as the minimal G-invariant subtree of T_ω. For the definition of the asymptotic cone see, for example, [11]. To see how to choose basepoints and scaling to ensure that the action is non-trivial see, for example, [7].

Proposition 6.2 (Exercise 19 in [2]) *Consider f_i-equivariant resolutions*

$$\phi_i : \tilde{K} \to T_{\mathbb{F}}.$$

Suppose $\lim T_{f_i} = T$, $\lim \Lambda_{\phi_i} = \Lambda$ and the sequences $(|f_i|)$ and $(||\phi_i||_K)$ are comparable. Then there is a resolution that sends lifts of leaves of Λ to points of T and is a Cantor function on edges of $\tilde{K}$.

Proof. A resolution $\phi : \tilde{K} \to T$ is determined by a choice of $\phi(\tilde{v})$ for a lift $\tilde{v}$ of each vertex v of K.

First, define a resolution $\phi' : \tilde{K} \to T_\omega$ by setting $\phi'(\tilde{v}) = [\phi_i(\tilde{v})]$. Since $(|f_i|)$ and $(||\phi_i||_K)$ are comparable, $\phi'(\tilde{v})$ is a valid point of T_ω. The resolution ϕ' maps leaves of Λ to points, and is a Cantor function on edges. However, T_ω is far from minimal. Let $\pi : T_\omega \to T$ be closest-point projection to the minimal invariant subtree, which is equivariantly isomorphic to T. Now let $\phi = \pi \circ \phi'$'; this is a resolution that still maps leaves of Λ to points, and is a Cantor function on edges, as required. $\square$

6.3 Finding kernel elements carried by leaves

Exercise 20 of [2] relies heavily on the results of [1]. The most important result is a structure theorem for resolutions of stable actions on real trees, summarized in the following theorem.

Theorem 6.3 (Theorems 9.4 and 9.5 of [1]) *Let Λ be a lamination on a 2-complex K, resolving a stable action of $G = \pi_1(K)$ on a real tree T. Then*

$$\Lambda = \Lambda_1 \sqcup \ldots \sqcup \Lambda_k.$$

Each component has a standard neighbourhood N_i carrying a subgroup $H_i \subset G$. Let T_i be the minimal H_i-invariant subtree of T. Each component is of one of the following types.

1. ***Surface type**. N_i is a cone-type 2-orbifold, with some annuli attached. H_i fits into a short exact sequence*

   $$1 \to \ker T_i \to H_i \to \pi_1(O) \to 1$$

   where O is a cone-type 2-orbifold.

2. ***Toral type**. T_i is a line, and H_i fits into a short exact sequence*

   $$1 \to \ker T_i \to H_i \to A \to 1$$

   where $A \subset \mathrm{Isom}(\mathbb{R})$.

3. ***Thin type**. H_i splits over an arc stabilizer, carried by a leaf of Λ_i.*

4. ***Simplicial type***. *All the leaves of Λ_i are compact, and N_i is an interval bundle over a leaf. H_i fits into a short exact sequence satisfying*

$$1 \to \ker T_i \to H_i \to C \to 1$$

where C is finite.

Furthermore, if E is a subgroup carried by a leaf, E fits into a short exact sequence of the form

$$1 \to \kappa \to E \to C \to 1$$

where κ fixes an arc of T and C is finite or cyclic.

In particular, the standard neighbourhoods induce a graph-of-spaces decomposition for K, and a corresponding graph-of-groups decomposition for G. The vertex spaces are the N_i and the closures of the components of $K - \cup_i N_i$. The edge spaces are boundary components of the N_i, and are all contained in a leaf. See theorem 5.13 of [1].

The proof of this exercise will also make use of the following result.

Proposition 6.4 (Corollary 5.9 of [1]) *If $h \in H_i$ fixes an arc of T_i then $h \in \ker T_i$.*

We are now ready to prove the exercise.

Theorem 6.5 (Exercise 20 of [2]) *In the situation of the exercise, the lamination Λ has a leaf carrying non-trivial elements of the kernel.*

Proof. Note that $G/\ker T$ is a limit group. Suppose no elements of $\ker T$ are carried by a leaf of Λ.

Consider Γ the graph of groups for G induced by Λ. The aim is to show that Γ really is a GAD. Since a GAD decomposition can be used to shorten, this contradicts the assumption that the f_i are short. We deal with each sort of vertex in turn.

1. Suppose Λ_i is of surface type. Then N_i is a cone-type 2-orbifold, with some annuli attached. Suppose $g \in H_i$ is carried by an annulus. Then g fixes an arc of T_i, so by proposition 6.4, $g \in \ker T_i$. But T_i contains a tripod, and tripod stabilizers are trivial, so $g \in \ker T$ contradicting the assumption. Therefore N_i can be assumed to have no attached annuli. Consider an element $g \in H_i$ carried by the leaf corresponding to a cone-point. Then g has finite order, so $g \in \ker T$, since $G/\ker T$ is a limit group. This contradicts the assumption, so N_i has no cone-points. Therefore N_i is genuinely a surface. Moreover, N_i carries a pseudo-Anosov homeomorphism, since it carries a minimal lamination.

2. If Λ_i is toral, then H_i is an extension

$$1 \to \ker T_i \to H_i \to A \to 1$$

for $A \subset \mathrm{Isom}\mathbb{R}$. The elements of $\ker T_i$ are carried by annuli in N_i. But $\ker T_i$ itself fits into an exact sequence

$$1 \to \kappa \to \ker T_i \to A' \to 1$$

where $\kappa \subset \ker T$ and A' is abelian. In order not to contradict the assumption that no elements of the kernel are carried by a leaf, therefore, κ must be trivial; so we have

$$1 \to A' \to H_i \to A \to 1.$$

and H_i acts faithfully on T. In particular, H_i embeds in the limit group $G/\ker T$, and so is a limit group. But A' is normal; since limit groups are torsion-free and CSA, it follows that H_i is free abelian.

3. If Λ_i is thin, then G splits over a subgroup H fixing an arc of T_i. By proposition 6.4, $H \subset \ker T_i$. But T_i contains a tripod, and tripod stabilizers are trivial, so $H \subset \ker T$; since H is carried by a leaf, H must be trivial by assumption. But this contradicts the assumption that G is freely indecomposable.

4. If Λ_i is simplicial, then H_i fits into the short exact sequence

$$1 \to \ker T_i \to H_i \to C \to 1.$$

for finite C. As in the toral case, the assumption implies that $\ker T_i$ is abelian, and H_i embeds in $G/\ker T$, and so is a limit group. But, again, H_i is torsion-free and CSA; so C is trivial, and H_i is abelian and fixes an arc of T.

Now consider an edge-group E of Γ. Then E is carried by a leaf, and satisfies

$$1 \to \kappa \to E \to C \to 1$$

where C is cyclic and κ fixes an arc of T. Then κ fits into a short exact sequence

$$1 \to \kappa' \to \kappa \to A \to 1$$

where $\kappa' \subset \ker T$ and A is abelian. By assumption, therefore, κ' is trivial and κ is abelian; furthermore, E acts faithfully on T, so embeds in $G/\ker T$ and is a limit group. Therefore E is free abelian.

In conclusion, Γ is a GAD. Just as in the proof of theorem 5.1, this contradicts the assumption that the f_i are all short. $\square$

6.4 Examples of limit groups

To complete their argument, Bestvina and Feighn need some elementary examples of limit groups. This theorem is required.

Theorem 6.6 (Exercise 21 of [2]) *Let Δ be a 1-edged GAD of a group G with a homomorphism q to a limit group Γ. Suppose:*

1. *the vertex groups of Δ are non-abelian,*
2. *the edge group of Δ is maximal abelian in each vertex group, and*
3. *q is injective on vertex groups of Δ. Then G is a limit group.*

This theorem is just a special case of proposition 4.22.

References

[1] M. Bestvina and M. Feighn. Stable actions of groups on real trees. *Invent. Math.*, 121, 1995.

[2] M. Bestvina and M. Feighn. Notes on Sela's work: Limit groups and Makanin-Razborov diagrams, 2003. Preprint.

[3] J. Bochnak, M. Coste, and M-F. Roy. *Géométrie algébrique réelle*, volume 3 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1987.

[4] Andrew J. Casson and Steven A. Bleiler. *Automorphisms of surfaces after Nielsen and Thurston*. Number 9 in London Mathematical Society Student Texts. Cambridge University Press, 1988.

[5] R. S. Lyndon. The equation $a^2b^2 = c^2$ in free groups. *Michigan Math. J*, 6, 1959.

[6] J. Morgan. Ergodic theory and free actions of groups on R-trees. *Invent. Math.*, 94, 1988.

[7] F. Paulin. Topologie de Gromov équivariante, structures hyperboliques et arbes réels. *Invent. Math.*, 94(1), 1988.

[8] E. Rips and Z. Sela. Structure and rigidity in hyperbolic groups I. *GAFA*, 4(3), 1994.

[9] Peter Scott and Terry Wall. Topological methods in group theory. In *Proc. Sympos., Durham, 1977*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge University Press, 1979.

[10] Z. Sela. Diophantine geometry over groups I: Makanin-Razborov diagrams. *Publ. Inst. Hautes Études Sci.*, 93, 2001.

[11] L. van den Dries and A. Wilkie. On Gromov's theorem concerning groups of polynomial growth and elementary logic. *J. Algebra*, 89, 1984.