Notes on Sela's work: Limit groups and Makanin-Razborov diagrams

Mladen Bestvina* Mark Feighn*

May 16, 2009

Abstract

This is the first in a planned series of papers giving an alternate approach to Zlil Sela's work on the Tarski problems. The present paper is an exposition of work of Kharlampovich-Myasnikov and Sela giving a parametrization of $Hom(G, \mathbb{F})$ where G is a finitely generated group and $\mathbb{F}$ is a non-abelian free group.

Contents

*The authors gratefully acknowledge support of the National Science Foundation. This material is based upon work supported by the National Science Foundation under Grants Nos. DMS-0502441 and DMS-0805440.

1 The Main Theorem

1.1 Introduction

This is the first of a planned series of papers giving an alternative approach to Zlil Sela's work on the Tarski problems [35, 34, 36, 38, 37, 39, 40, 41, 31, 32]. The present paper is an exposition of the following result of Kharlampovich-Myasnikov [14, 15] and Sela [34]:

Theorem. *Let G be a finitely generated non-free group. There is a finite collection $\{q_i : G \to \Gamma_i\}$ of proper epimorphisms of G such that, for any homomorphism f from G to a free group F, there is $\alpha \in Aut(G)$ such that $f\alpha$ factors through some q_i.*

A more refined statement is given in the Main Theorem on page 7. Our approach, though similar to Sela's, differs in several aspects: notably a different measure of complexity and a more geometric proof which avoids the use of the full Rips theory for finitely generated groups acting on $\mathbb{R}$-trees; see Section 7. We attempted to include enough background material to make the paper self-contained. See Paulin [24] and Champetier-Guirardel [5] for accounts of some of Sela's work on the Tarski problems.

The first version of these notes was circulated in 2003. In the meantime Henry Wilton [45] made available solutions to the exercises in the notes. We also thank Wilton for making numerous comments that led to many improvements.

Remark 1.1. In the theorem above, since G is finitely generated we may assume that F is also finitely generated. If F is abelian, then any f factors through the abelianization of G mod its torsion subgroup and we are in the situation of Example 1.4 below. Finally, if F_1 and F_2 are finitely generated non-abelian free groups then there is an injection $F_1 \to F_2$. So, if $\{q_i\}$ is a set of epimorphisms that satisfies the conclusion of the theorem for maps to F_2, then $\{q_i\}$ also works for maps to F_1. Therefore, throughout the paper we work with a fixed finitely generated non-abelian free group $\mathbb{F}$.

Notation 1.2. Finitely generated (finitely presented) is abbreviated fg (respectively fp).

The main goal of [34] is to give an answer to the following:

Question 1. *Let G be an fg group. Describe the set of all homomorphisms from G to $\mathbb{F}$.*

Example 1.3. When G is a free group, we can identify $Hom(G, \mathbb{F})$ with the cartesian product $\mathbb{F}^n$ where $n = rank(G)$.

Example 1.4. If $G = \mathbb{Z}^n$, let $\mu : \mathbb{Z}^n \to \mathbb{Z}$ be the projection to one of the coordinates. If $h : \mathbb{Z}^n \to \mathbb{F}$ is a homomorphism, there is an automorphism $\alpha : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $h\alpha$ factors through μ. This provides an explicit (although not 1-1) parametrization of $Hom(G, \mathbb{F})$ by $Aut(\mathbb{Z}^n) \times Hom(\mathbb{Z}, \mathbb{F}) \cong GL_n(\mathbb{Z}) \times \mathbb{F}$.

Example 1.5. When G is the fundamental group of a closed genus g orientable surface, let $\mu : G \to F_g$ denote the homomorphism to a free group of rank g induced by the (obvious) retraction of the surface to the rank g graph. It is a folk theorem[1] that for every homomorphism $f : G \to \mathbb{F}$ there is an automorphism $\alpha : G \to G$ (induced by a homeomorphism of the surface) so that $f\alpha$ factors through μ. The theorem was generalized to the case when G is the fundamental group of a non-orientable closed surface by Grigorchuk and Kurchanov [9]. Interestingly, in this generality the single map μ is replaced by a finite collection $\{\mu_1, \cdots, \mu_k\}$ of maps from G to a free group F. In other words, for all $f \in Hom(G, \mathbb{F})$ there is $\alpha \in Aut(G)$ induced by a homeomorphism of the surface such that $f\alpha$ factors through some μ_i.

1.2 Basic properties of limit groups

Another goal is to understand the class of groups that naturally appear in the answer to the above question, these are called limit groups.

Definition 1.6. Let G be an fg group. A sequence $\{f_i\}$ in $Hom(G, \mathbb{F})$ is *stable* if, for all $g \in G$, the sequence $\{f_i(g)\}$ is eventually always 1 or eventually never 1. The *stable kernel* of $\{f_i\}$, denoted $\underrightarrow{Ker}\, f_i$, is

$$\{g \in G \mid f_i(g) = 1 \text{ for almost all } i\}.$$

An fg group Γ is a *limit group* if there is an fg group G and a stable sequence $\{f_i\}$ in $Hom(G, \mathbb{F})$ so that $\Gamma \cong G/\underrightarrow{Ker}\, f_i$.

Remark 1.7. One can view each f_i as inducing an action of G on the Cayley graph of $\mathbb{F}$, and then can pass to a limiting $\mathbb{R}$-tree action (after a subsequence). If the limiting tree is not a line, then $\underrightarrow{Ker}\, f_i$ is precisely the kernel of this action and so Γ acts faithfully. This explains the name.

[1] see Zieschang [46] and Stallings [43]

Definition 1.8. An fg group Γ is *residually free* if for every element $\gamma \in \Gamma$ there is $f \in Hom(\Gamma, \mathbb{F})$ such that $f(\gamma) \neq 1$. It is *ω-residually free* if for every finite subset $X \subset \Gamma$ there is $f \in Hom(\Gamma, \mathbb{F})$ such that $f|X$ is injective.

Exercise 2. *Residually free groups are torsion free.*

Exercise 3. *Free groups and free abelian groups are ω-residually free.*

Exercise 4. *The fundamental group of $n\mathbb{P}^2$ for $n = 1$, 2, or 3 is not ω-residually free, see [18].*

Exercise 5. *Every ω-residually free group is a limit group.*

Exercise 6. *An fg subgroup of an ω-residually free group is ω-residually free.*

Exercise 7. *Every non-trivial abelian subgroup of an ω-residually free group is contained in a unique maximal abelian subgroup. For example, $F \times \mathbb{Z}$ is not ω-residually free for any non-abelian F.*

Lemma 1.9. *Let $G_1 \to G_2 \to \cdots$ be an infinite sequence of epimorphisms between fg groups. Then the sequence*

$$Hom(G_1, \mathbb{F}) \leftarrow Hom(G_2, \mathbb{F}) \leftarrow \cdots$$

eventually stabilizes (consists of bijections).

Proof. Embed $\mathbb{F}$ as a subgroup of $SL_2(\mathbb{R})$. That the corresponding sequence of varieties $Hom(G_i, SL_2(\mathbb{R}))$ stabilizes follows from algebraic geometry, and this proves the lemma. □

Corollary 1.10. *A sequence of epimorphisms between ($\omega-$)residually free groups eventually stabilizes.* □

Lemma 1.11. *Every limit group is ω-residually free.*

Proof. Let Γ be a limit group, and let G and $\{f_i\}$ be as in the definition. Without loss, G is fp. Now consider the sequence of quotients

$$G \to G_1 \to G_2 \to \cdots \to \Gamma$$

obtained by adjoining one relation at a time. If Γ is fp the sequence terminates, and in general it is infinite. Let $G' = G_j$ be such that $Hom(G', \mathbb{F}) = Hom(\Gamma, \mathbb{F})$. All but finitely many f_i factor through G' since each added relation is sent to 1 by almost all f_i. It follows that these f_i factor through Γ and each non-trivial element of Γ is sent to 1 by only finitely many f_i. By definition, Γ is ω-residually free. □

The next two exercises will not be used in this paper but are included for their independent interest.

Exercise 8. *Every ω-residually free group Γ embeds into $PSL_2(\mathbb{R})$, and also into $SO(3)$.*

Exercise 9. *Let Γ be ω-residually free. For any finite collection of nontrivial elements $g_1, \cdots, g_k \in \Gamma$ there is an embedding $\Gamma \to PSL_2(\mathbb{R})$ whose image has no parabolic elements and so that $g_1, \cdots, g_k$ go to hyperbolic elements.*

1.3 Modular groups and the statement of the main theorem

Only certain automorphisms, called *modular automorphisms*, are needed in the theorem on page 2. This section contains a definition of these automorphisms.

Definition 1.12. Free products with amalgamations and HNN-decompositions of a group G give rise to *Dehn twist automorphisms of G.* Specifically, if $G = A *_C B$ and if z is in the centralizer $Z_B(C)$ of C in B, then the automorphism α_z of G, called the *Dehn twist in z*, is determined as follows.

$$\alpha_z(g) = \begin{cases} g, & \text{if } g \in A; \\ zgz^{-1}, & \text{if } g \in B. \end{cases}$$

If $C \subset A$, $\phi : C \to A$ is a monomorphism, $G = A*_C = \langle A, t \mid tat^{-1} = \phi(a), a \in A\rangle$,[2] and $z \in Z_A(C)$, then α_z is determined as follows.

$$\alpha_z(g) = \begin{cases} g, & \text{if } g \in A; \\ gz, & \text{if } g = t. \end{cases}$$

Definition 1.13. A GAD[3] of a group G is a finite graph of groups decomposition[4] of G with abelian edge groups in which some of the vertices are designated QH[5] and some others are designated *abelian*, and the following holds.

[2] t is called a *stable letter.*

[3] Generalized Abelian Decomposition

[4] We will use the terms *graph of groups decomposition* and *splitting* interchangeably. Without further notice, splittings are always *minimal*, i.e. the associated G-tree has no proper invariant subtrees.

[5] Quadratically Hanging

- A QH-vertex group is the fundamental group of a compact surface S with boundary and the boundary components correspond to the incident edge groups (they are all infinite cyclic). Further, S carries a pseudoAnosov homeomorphism (so S is a torus with 1 boundary component or $\chi(S) \leq -2$).

- An abelian vertex group A is non-cyclic abelian. Denote by $P(A)$ the subgroup of A generated by incident edge groups. The *peripheral subgroup of* A, denoted $\overline{P}(A)$, is the subgroup of A that dies under every homomorphism from A to $\mathbb{Z}$ that kills $P(A)$, i.e.

$$\overline{P}(A) = \cap\{Ker(f) \mid f \in Hom(A, \mathbb{Z}), P(A) \subset Ker(f)\}.$$

The non-abelian non-QH vertices are *rigid*.

Remark 1.14. We allow the possibility that edge and vertex groups of GAD's are not fg.

Remark 1.15. If Δ is a GAD for a fg group G, and if A is an abelian vertex group of Δ, then there are epimorphisms $G \to A/P(A) \to A/\overline{P}(A)$. Hence, $A/P(A)$ and $A/\overline{P}(A)$ are fg. Since $A/\overline{P}(A)$ is also torsion free, $A/\overline{P}(A)$ is free, and so $A = A_0 \oplus \overline{P}(A)$ with $A_0 \cong A/\overline{P}(A)$ a retract of G. Similarly, $A/\overline{P}(A)$ is a direct summand of $A/P(A)$. A summand complementary to $A/\overline{P}(A)$ in $A/P(A)$ must be a torsion group by the definition of $\overline{P}(A)$. In particular, $P(A)$ has finite index in $\overline{P}(A)$. It also follows from the definition of $\overline{P}(A)$ that any automorphism leaving $P(A)$ invariant must leave $\overline{P}(A)$ invariant as well. It follows that if A is torsion free, then any automorphism of A that is the identity when restricted to $P(A)$ is also the identity when restricted to $\overline{P}(A)$.

Definition 1.16. The *modular group* $Mod(\Delta)$ associated to a GAD Δ of G is the subgroup of $Aut(G)$ generated by

- inner automorphisms of G,

- Dehn twists in elements of G that centralize an edge group of Δ,

- unimodular[6] automorphisms of an abelian vertex group that are the identity on its peripheral subgroup and all other vertex groups, and

[6]The induced automorphism of $A/\overline{P}(A)$ has determinant 1.

- automorphisms induced by homeomorphisms of surfaces S underlying QH-vertices that fix all boundary components. If S is closed and orientable, we require the homeomorphisms to be orientation-preserving[7].

The *modular group of* G, denoted $Mod(G)$, is the subgroup of $Aut(G)$ generated by $Mod(\Delta)$ for all GAD's Δ of G. At times it will be convenient to view $Mod(G)$ as a subgroup of $Out(G)$. In particular, we will say that an element of $Mod(G)$ is *trivial* if it is an inner automorphism.

Definition 1.17. A *generalized Dehn twist* is a Dehn twist or an automorphism α of $G = A *_C B$ or $G = A*_C$ where in each case A is abelian, α restricted to $\overline{P}(A)$ and B is the identity, and α induces a unimodular automorphism of $A/\overline{P}(A)$. Here $\overline{P}(A)$ is the peripheral subgroup of A when we view $A *_C B$ or $G = A$ as a GAD with one or zero edges and abelian vertex A. If C is an edge groups of a GAD for G and if $z \in Z_G(C)$, then C determines a splitting of G as above and so also a Dehn twist in z. Similarly, an abelian vertex A of a GAD determines[8] a splitting $A *_C B$ and so also generalized Dehn twists.

Exercise 10. *$Mod(G)$ is generated by inner automorphisms together with generalized Dehn twists.*

Definition 1.18. A *factor set* for a group G is a finite collection of proper epimorphisms $\{q_i : G \to G_i\}$ such that if $f \in Hom(G, \mathbb{F})$ then there is $\alpha \in Mod(G)$ such that $f\alpha$ factors through some q_i.

Main Theorem ([14, 15, 35])**.** *Let G be an fg group that is not free. Then, G has a factor set $\{q_i : G \to \Gamma_i\}$ with each Γ_i a limit group. If G is not a limit group, we can always take α to be the identity.*

We will give two proofs–one in Section 4 and the second, which uses less in the way of technical machinery, in Section 7. In the remainder of this section, we explore some consequences of the Main Theorem and then give another description of limit groups.

1.4 Makanin-Razborov diagrams

Corollary 1.19. *Iterating the construction of the Main Theorem (for Γ_i's etc.) yields a finite tree of groups terminating in groups that are free.*

[7]We will want our homeomorphisms to be products of Dehn twists.
[8]by folding together the edges incident to A

Proof. If $\Gamma \to \Gamma'$ is a proper epimorphism between limit groups, then since limit groups are residually free, $Hom(\Gamma', \mathbb{F}) \subsetneq Hom(\Gamma, \mathbb{F})$. We are done by Lemma 1.9. □

Definition 1.20. The tree of groups and epimorphisms provided by Corollary 1.19 is called an *MR-diagram*[9] for G (with respect to $\mathbb{F}$). If

$$G \xrightarrow{q} \Gamma_1 \xrightarrow{q_1} \Gamma_2 \xrightarrow{q_2} \cdots \xrightarrow{q_{m-1}} \Gamma_m$$

is a branch of an MR-diagram and if $f \in Hom(G, \mathbb{F})$ then we say that f *MR-factors* through this branch if there are $\alpha \in Mod(G)$ (which is the identity if G is not a limit group), $\alpha_i \in Mod(\Gamma_i)$, for $1 \leq i < m$, and $f' \in Hom(\Gamma_m, \mathbb{F})$ (recall Γ_m is free) such that $f = f' q_{m-1} \alpha_{m-1} \cdots q_1 \alpha_1 q \alpha$.

Remark 1.21. The key property of an MR-diagram for G is that, for $f \in Hom(G, \mathbb{F})$, there is a branch of the diagram through which f MR-factors. This provides an answer to Question 1 in that $Hom(G, \mathbb{F})$ is parametrized by branches of an MR-diagram and, for each branch as above, $Mod(G) \times Mod(\Gamma_1) \times \cdots \times Mod(\Gamma_{m-1}) \times Hom(\Gamma_m, \mathbb{F})$. Note that if Γ_m has rank n, then $Hom(\Gamma_m, \mathbb{F}) \cong \mathbb{F}^n$.

In [32], Sela constructed MR-diagrams with respect to hyperbolic groups. In her thesis [1], Emina Alibegović constructed MR-diagrams with respect to limit groups. More recently, Daniel Groves [10, 11] constructed MR-diagrams with respect to torsion-free groups that are hyperbolic relative to a collection of free abelian subgroups.

1.5 Abelian subgroups of limit groups

Corollary 1.22. *Abelian subgroups of limit groups are fg and free.*

Along with the Main Theorem, the proof of Corollary 1.22 will depend on an exercise and two lemmas.

Exercise 11 ([34, Lemma 2.3])**.** *Let M be a non-cyclic maximal abelian subgroup of the limit group Γ.*

1. *If $\Gamma = A *_C B$ with C abelian, then M is conjugate into A or B.*

[9] for Makanin-Razborov, cf. [19, 20, 25].

2. *If $\Gamma = A*_C$ with C abelian, then either M is conjugate into A or there is a stable letter t such that M is conjugate to $M' = \langle C, t\rangle$ and $\Gamma = A *_C M'$.*

As a consequence, if $\alpha \in Mod(\Gamma)$ is a generalized Dehn twist and $\alpha|M$ is non-trivial, then there is an element $\gamma \in \Gamma$ and a GAD *$\Delta = M *_C B$ or $\Delta = M$ for Γ such that, up to conjugation by γ, α is induced by a unimodular automorphism of $M/\overline{P}(M)$ (as in Definition 1.17). (Hint: Use Exercise 7.)*

Lemma 1.23. *Suppose that Γ is a limit group with factor set $\{q_i : \Gamma \to G_i\}$. If H is a (not necessarily fg) subgroup of Γ such that, for every homomorphism $f : \Gamma \to \mathbb{F}$, $f|H$ factors through some $q_i|H$ (pre-compositions by automorphisms of Γ not needed) then, for some i, $q_i|H$ is injective.*

Proof. Suppose not and let $1 \neq h_i \in Ker(q_i|H)$. Since Γ is a limit group, there is $f \in Hom(\Gamma, \mathbb{F})$ that is injective on $\{1, h_1, \cdots, h_n\}$. On the other hand, $f|H$ factors through some $q_i|H$ and so $h_i = 1$, a contradiction. □

Lemma 1.24. *Let M be a non-cyclic maximal abelian subgroup of the limit group Γ. There is an epimorphism $r : \Gamma \to A$ where A is free abelian and every modular automorphism of Γ is trivial*[10] *when restricted to $M \cap Ker(r)$.*

Proof. By Exercise 10, it is enough to find r such that $\alpha|M \cap Ker(r)$ is trivial for every generalized Dehn twist $\alpha \in Mod(\Gamma)$. By Exercise 11 and Remark 1.15, there is a fg free abelian subgroup M_α of M and a retraction $r_\alpha : \Gamma \to M_\alpha$ such that $\alpha|M \cap Ker(r_\alpha)$ is trivial. Let $r = \Pi_\alpha r_\alpha : \Gamma \to \Pi_\alpha M_\alpha$ and let A be the image of r. Since Γ is fg, so is A. Hence A is free abelian. □

Proof of Corollary 1.22. Let M be a maximal abelian subgroup of a limit group Γ. We may assume that M is not cyclic. Since Γ is torsion free, it is enough to show that M is fg. By restricting the map r of Lemma 1.24 to M, we see that $M = A \oplus A'$ where A is fg and each $\alpha|A'$ is trivial. Let $\{q_i : \Gamma \to \Gamma_i\}$ be a factor set for Γ given by Theorem 1.3. By Lemma 1.23, A' injects into some Γ_i. Since $Hom(\Gamma_i, \mathbb{F}) \subsetneq Hom(\Gamma, \mathbb{F})$, we may conclude by induction that A' and hence M is fg. □

[10] agrees with the restriction of an inner automorphism of Γ.

1.6 Constructible limit groups

It will turn out that limit groups can be built up inductively from simpler limit groups. In this section, we give this description and list some properties that follow.

Definition 1.25. We define a hierarchy of fg groups – if a group belongs to this hierarchy it is called a CLG[11].

Level 0 of the hierarchy consists of fg free groups.

A group Γ belongs to level $\leq n+1$ iff either it has a free product decomposition $\Gamma = \Gamma_1 * \Gamma_2$ with Γ_1 and Γ_2 of level $\leq n$ or it has a homomorphism $\rho : \Gamma \to \Gamma'$ with Γ' of level $\leq n$ and it has a GAD such that

- ρ is injective on the peripheral subgroup of each abelian vertex group.
- ρ is injective on each edge group E and at least one of the images of E in a vertex group of the one-edged splitting induced by E is a maximal abelian subgroup.
- The image of each QH-vertex group is a non-abelian subgroup of Γ'.
- For every rigid vertex group B, ρ is injective on the *envelope* $\tilde{B}$ *of* B, defined by first replacing each abelian vertex with the peripheral subgroup and then letting $\tilde{B}$ be the subgroup of the resulting group generated by B and by the centralizers of incident edge-groups.

Example 1.26. A fg free abelian group is a CLG of level one (consider a one-point GAD for $\mathbb{Z}^n$ and $\rho : \mathbb{Z}^n \to \langle 0 \rangle$). The fundamental group of a closed surface S with $\chi(S) \leq -2$ is a CLG of level one. For example, an orientable genus 2 surface is a union of 2 punctured tori and the retraction to one of them determines ρ. Similarly, a non-orientable genus 2 surface is the union of 2 punctured Klein bottles.

Example 1.27. Start with the circle and attach to it 3 surfaces with one boundary component, with genera 1, 2, and 3 say. There is a retraction to the surface of genus 3 that is the union of the attached surfaces of genus 1 and 2. This retraction sends the genus 3 attached surface say to the genus 2 attached surface by "pinching a handle". The GAD has a central vertex labeled $\mathbb{Z}$ and there are 3 edges that emanate from it, also labeled $\mathbb{Z}$. Their other endpoints are QH-vertex groups. The map induced by retraction satisfies the requirements so the fundamental group of the 2-complex built is a CLG.

[11] Constructible Limit Group

Example 1.28. Choose a primitive[12] w in the fg free group F and form $\Gamma = F *_{\mathbb{Z}} F$, the *double of F along $\langle w \rangle$* (so $1 \in \mathbb{Z}$ is identified with w on both sides). There is a retraction $\Gamma \to F$ that satisfies the requirements (both vertices are rigid), so Γ is a CLG.

The following can be proved by induction on levels.

Exercise 12. *Every CLG is fp, in fact coherent. Every fg subgroup of a CLG is a CLG. (Hint: a graph of coherent groups over fg abelian groups is coherent.)*

Exercise 13. *Every abelian subgroup of a CLG Γ is fg and free, and there is a uniform bound to the rank. There is a finite $K(\Gamma, 1)$.*

Exercise 14. *Every non-abelian, freely indecomposable CLG admits a* principal splitting *over $\mathbb{Z}$: $A *_{\mathbb{Z}} B$ or $A *_{\mathbb{Z}}$ with A, B non-cyclic, and in the latter case $\mathbb{Z}$ is maximal abelian in the whole group.*

Exercise 15. *Every CLG is ω-residually free.*

The last exercise is more difficult than the others. It explains where the conditions in the definition of CLG come from. The idea is to construct homomorphisms $G \to \mathbb{F}$ by choosing complicated modular automorphisms of G, composing with ρ and then with a homomorphism to $\mathbb{F}$ that comes from the inductive assumption.

Example 1.29. Consider an index 2 subgroup H of an fg free group F and choose $g \in F \setminus H$. Suppose that $G := H *_{\langle g^2 \rangle} \langle g \rangle$ is freely indecomposable and admits no principal cyclic splitting. There is the obvious map $G \to F$, but G is not a limit group (Exercise 14 and Theorem 1.30). This shows the necessity of the last condition in the definition of CLG's. [13]

In Section 6, we will show:

Theorem 1.30. *For an fg group G, the following are equivalent.*

1. *G is a CLG.*

[12] no proper root

[13] The element $g := a^2b^2a^{-2}b^{-1} \notin H := \langle a, b^2, bab^{-1} \rangle \subset F := \langle a, b \rangle$ is such an example. This can be seen from the fact that if $\langle x, y, z \rangle$ denotes the displayed basis for H, then $g^2 = x^2yx^{-2}y^{-1}z^2yz^{-2}$ is Whitehead reduced and each basis element occurs at least 3 times.

2. *G is ω-residually free.*

3. *G is a limit group.*

The fact that ω-residually free groups are CLG's is due to O. Kharlampovich and A. Myasnikov [16]. Limit groups act freely on $\mathbb{R}^n$-trees; see Remeslennikov [27] and Guirardel [13]. Kharlampovich-Myasnikov [15] prove that limit groups act freely on $\mathbb{Z}^n$-trees where $\mathbb{Z}^n$ is lexicographically ordered. Remeslennikov [26] also demonstrated that 2-residually free groups are ω-residually free.

2 The Main Proposition

Definition 2.1. An fg group is *generic* if it is torsion free, freely indecomposable, non-abelian, and not a closed surface group.

The Main Theorem will follow from the next proposition.

Main Proposition. *Generic limit groups have factor sets.*

Before proving this proposition, we show how it implies the Main Theorem.

Definition 2.2. Let G and G' be fg groups. The minimal number of generators for G is denoted $\mu(G)$. We say that G is *simpler* than G' if there is an epimorphism $G' \to G$ and either $\mu(G) < \mu(G')$ or $\mu(G) = \mu(G')$ and $Hom(G, \mathbb{F}) \subsetneq Hom(G', \mathbb{F})$.

Remark 2.3. It follows from Lemma 1.9 that every sequence $\{G_i\}$ with G_{i+1} simpler than G_i is finite.

Definition 2.4. If G is an fg group, then by $RF(G)$ denote the *universal residually free quotient of* G, i.e. the quotient of G by the (normal) subgroup consisting of elements killed by every homomorphism $G \to \mathbb{F}$.

Remark 2.5. $Hom(G, \mathbb{F}) = Hom(RF(G), \mathbb{F})$ and for every proper quotient G' of $RF(G)$, $Hom(G', \mathbb{F}) \subsetneq Hom(G, \mathbb{F})$.

The Main Proposition implies the Main Theorem. Suppose that G is an fg group that is not free. By Remark 2.3, we may assume that the Main Theorem holds for groups that are simpler than G. By Remark 2.5, we may assume that G is residually free, and so also torsion free. Examples 1.4

and 1.5 show that the Main Theorem is true for abelian and closed surface groups. If $G = U * V$ with U non-free and freely indecomposable and with V non-trivial, then U is simpler than G. So, U has a factor set $\{q_i : U \to L_i\}$, and $\{q_i * Id_V : U * V \to L_i * V\}$ is a factor set for G.

If G is not a limit group, then there is a non-empty finite subset $\{g_i\}$ of G such that any homomorphism $G \to \mathbb{F}$ kills one of the g_i. We then have a factor set $\{G \to H_i := G/\langle\langle g_i \rangle\rangle\}$. Since $Hom(H_i, \mathbb{F}) \subsetneq Hom(G, \mathbb{F})$, by induction the Main Theorem holds for H_i and so for G.

If G is generic and a limit group, then the Main Proposition gives a factor set $\{q_i : G \to G_i\}$ for G. Since G is residually free, each G_i is simpler than G. We are assuming that the Main Theorem then holds for each G_i and this implies the result for G. □

3 Review: Measured laminations and $\mathbb{R}$-trees

The proof of the Main Proposition will use a theorem of Sela describing the structure of certain real trees. This in turn depends on the structure of measured laminations. In Section 7, we will give an alternate approach that only uses the lamination results. First these concepts are reviewed. A more leisurely review with references is [2].

3.1 Laminations

Definition 3.1. A *measured lamination* Λ on a simplicial 2-complex K consists of a closed subset $|\Lambda| \subset |K|$ and a *transverse measure* μ. $|\Lambda|$ is disjoint from the vertex set, intersects each edge in a Cantor set or empty set, and intersects each 2-simplex in 0, 1, 2, or 3 families of straight line segments spanning distinct sides. The measure μ assigns a non-negative number $\int_I \mu$ to every interval I in an edge whose endpoints are outside $|\Lambda|$. There are two conditions:

1. **(compatibility)** If two intervals I, J in two sides of the same triangle Δ intersect the same components of $|\Lambda| \cap \Delta$ then $\int_I \mu = \int_J \mu$.

2. **(regularity)** μ restricted to an edge is equivalent under a "Cantor function" to the Lebesgue measure on an interval in $\mathbb{R}$.

A path component of $|\Lambda|$ is a *leaf*.

Two measured laminations on K are considered equivalent if they assign the same value to each edge.

Proposition 3.2 (Morgan-Shalen [21])**.** *Let Λ be a measured lamination on compact K. Then*

$$\Lambda = \Lambda_1 \sqcup \cdots \sqcup \Lambda_k$$

so that each Λ_i is either minimal *(each leaf is dense in $|\Lambda_i|$) or* simplicial *(each leaf is compact, a regular neighborhood of $|\Lambda_i|$ is an I-bundle over a leaf and $|\Lambda_i|$ is a Cantor set subbundle).*

There is a theory, called the *Rips machine*, for analyzing minimal measured laminations. It turns out that there are only 3 qualities.

Example 3.3 (Surface type). Let S be a compact hyperbolic surface (possibly with totally geodesic boundary). If S admits a pseudoAnosov homeomorphism then it also admits *filling measured geodesic laminations* – these are measured laminations Λ (with respect to an appropriate triangulation) such that each leaf is a biinfinite geodesic and all complementary components are ideal polygons or crowns. Now to get the model for a general surface type lamination attach finitely many annuli $S^1 \times I$ with lamination $S^1 \times$ (Cantor set) to the surface along arcs transverse to the geodesic lamination. If these additional annuli do not appear then the lamination is of *pure surface type*. See Figure 1.

Example 3.4 (Toral type). Fix a closed interval $I \subset \mathbb{R}$, a finite collection of pairs (J_i, J_i') of closed intervals in I, and isometries $\gamma_i : J_i \to J_i'$ so that:

1. If γ_i is orientation reversing then $J_i = J_i'$ and the midpoint is fixed by γ_i.
2. The length of the intersection of all J_i and J_i' (over all i) is more than twice the translation length of each orientation preserving γ_i and the fixed points of all orientation reversing γ_i are in the middle third of the intersection.

Now glue a foliated band for each pair (J_i, J_i') so that following the band maps J_i to J_i' via γ_i. Finally, using Cantor functions blow up the foliation to a lamination. There is no need to explicitly allow adding annuli as in the surface case since they correspond to $\gamma_i = Id$. The subgroup of $Isom(\mathbb{R})$ generated by the extensions of the γ_i's is the *Bass group*. The lamination is minimal iff its Bass group is not discrete.

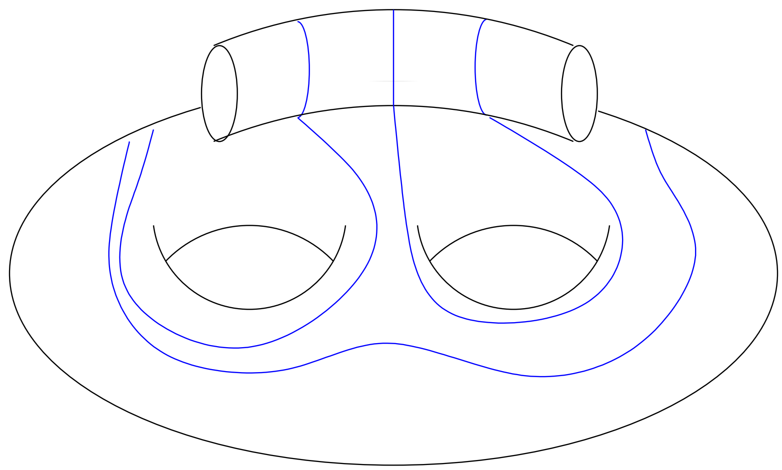

Figure 1: A surface with an additional annulus and some pieces of leaves.

Example 3.5 (Thin type). This is the most mysterious type of all. It was discovered by Gilbert Levitt, see [17]. In the *pure* case (no annuli attached) the leaves are 1-ended trees (so this type naturally lives on a 2-complex, not on a manifold). By performing certain moves (sliding, collapsing) that don't change the homotopy type (respecting the lamination) of the complex one can transform it to one that contains a (thin) band. This band induces a non-trivial free product decomposition of $\pi_1(K)$, assuming that the component is a part of a resolution of a tree (what's needed is that loops that follow leaves until they come close to the starting point and then they close up are non-trivial in π_1).

In the general case we allow additional annuli to be glued, just like in the surface case. Leaves are then 1-ended trees with circles attached.

Theorem 3.6 ("Rips machine")**.** *Let Λ be a measured lamination on a finite 2-complex K, and let Λ_i be a minimal component of Λ. There is a neighborhood N (we refer to it as a* standard *neighborhood) of $|\Lambda_i|$, a finite 2-complex N' with measured lamination Λ' as in one of 3 model examples, and there is a π_1-isomorphism $f : N \to N'$ such that $f^*(\Lambda') = \Lambda$.*

We refer to Λ_i as being of *surface*, *toral*, or *thin* type.

3.2 Dual trees

Let G be an fg group and let $\hat{K}$ be a simply connected 2-dimensional simplicial G-complex so that, for each simplex Δ of $\hat{K}$, $Stab(\Delta) = Fix(\Delta)$.[14] Let $\hat{\Lambda}$ be a G-invariant measured lamination in $\hat{K}$. There is an associated real G-tree $T(\hat{\Lambda})$ constructed as follows. Consider the pseudo-metric on $\hat{K}$ obtained by minimizing the $\hat{\Lambda}$-length of paths between points. The real tree $T(\hat{\Lambda})$ is the associated metric space[15]. There is a natural map $\hat{K} \to T(\hat{\Lambda})$ and we say that $(\hat{K}, \hat{\Lambda})$ is a *model* for $T(\hat{\Lambda})$ if

- for each edge $\hat{e}$ of $\hat{K}$, $T(\hat{\Lambda} \mid \hat{e}) \to T(\hat{\Lambda})$ is an isometry (onto its image) and
- the quotient $\hat{K}/G$ is compact.

If a tree T admits a model $(\hat{K}, \hat{\Lambda})$, then we say that T is *dual* to $(\hat{K}, \hat{\Lambda})$. This is denoted $T = Dual(\hat{K}, \hat{\Lambda})$. We will use the quotient $(K, \Lambda) := (\hat{K}, \hat{\Lambda})/G$ with simplices decorated (or labeled) with stabilizers to present a model and sometimes abuse notation by calling (K, Λ) a model for T.

Remark 3.7. Often the G-action on $\hat{K}$ is required to be free. We have relaxed this condition in order to be able to consider actions of fg groups. For example, if T is a minimal[16], simplicial G-tree (with the metric where edges have length one[17]) then there is a lamination $\hat{\Lambda}$ in T such that $Dual(T, \hat{\Lambda}) = T$.[18]

If S and T are real G-trees, then an equivariant map $f : S \to T$ is a *morphism* if every compact segment of S has a finite partition such that the restriction of f to each element is an isometry or trivial[19].

If S is a real G-tree with G fp, then there is a real G-tree T with a model and a morphism $f : T \to S$. The map f is obtained by constructing an equivariant map to S from the universal cover of a 2-complex with fundamental group G. In general, if $(\hat{K}, \hat{\Lambda})$ is a model for T and if $T \to S$ is a morphism then the composition $\hat{K} \to T \to S$ is a *resolution* of S.

[14] $Stab(\Delta) := \{g \in G \mid g\Delta = \Delta\}$ and $Fix(\Delta) := \{g \in G \mid gx = x, x \in \Delta\}$

[15] identify points of pseudo-distance 0

[16] no proper invariant subtrees

[17] This is called the *simplicial metric* on T.

[18] The metric and simplicial topologies on T don't agree unless T is locally finite. But, the action of G is by isomorphisms in each structure. So, we will be sloppy and ignore this distinction.

[19] has image a point

3.3 The structure theorem

Here we discuss a structure theorem (see Theorem 3.13) of Sela for certain actions of an fg torsion free group G on real trees. The actions we consider will usually be super stable[20], have primitive[21] abelian (non-degenerate) arc stabilizers, and have trivial tripod[22] stabilizers. There is a short list of basic examples.

Example 3.8 (Pure surface type). A real G-tree T is *of pure surface type* if it is dual to the universal cover of (K, Λ) where K is a compact surface and Λ is of pure surface type. We will usually use the alternate model where boundary components are crushed to points and are labeled $\mathbb{Z}$.

Example 3.9 (Linear). The tree T is *linear* if G is abelian, T is a line and there an epimorphism $G \to \mathbb{Z}^n$ such that G acts on T via a free $\mathbb{Z}^n$-action on T. In particular, T is dual to $(\hat{K}, \hat{\Lambda})$ where $\hat{K}$ is the universal cover of the 2-skeleton of an n-torus K. For simplicity, we often complete K with its lamination to the whole torus. This is a special case of a toral lamination.

Example 3.10 (Pure thin). The tree T is *pure thin* if it is dual to the universal cover of a finite 2-complex K with a pure thin lamination Λ. If T is pure thin then $G \cong F * V_1 * \cdots * V_m$ where F is non-trivial and fg free and $\{V_1, \cdots, V_m\}$ represents the conjugacy classes of non-trivial point stabilizers in T.

Example 3.11 (Simplicial). The tree T is *simplicial* if it is dual to $(\hat{K}, \hat{\Lambda})$ where all leaves of $\Lambda := \hat{\Lambda}/G$ are compact. If T is simplicial it is convenient to crush the leaves and complementary components to points in which case $\hat{K}$ becomes a tree isomorphic to T.

If $\mathcal{K}$ is a graph of 2-complexes with underlying graph of groups $\mathcal{G}$[23] then there is a simplicial $\pi_1(\mathcal{G})$-space $\hat{K}(\mathcal{K})$ obtained by gluing copies of $\hat{K}_e \times I$ and $\hat{K}_v$'s equipped with a simplicial $\pi_1(\mathcal{G})$-map $\hat{K}(\mathcal{K}) \to T(\mathcal{G})$ that crushes to points copies of $\hat{K}_e \times \{point\}$ as well as the $\hat{K}_v$'s.

Definition 3.12. A real G-tree is *very small* if the action is non-trivial[24], minimal, the stabilizers of non-degenerate arcs are primitive abelian, and the stabilizers of non-degenerate tripods are trivial.

---

[20] If $J \subset I$ are (non-degenerate) arcs in T and if $Fix_T(I)$ is non-trivial, then $Fix_T(J) = Fix_T(I)$

[21] root-closed

[22] a cone on 3 points

[23] for each bonding map $\phi_e : G_e \to G_v$ there are simplicial G_e- and G_v-complexes $\hat{K}_e$ and $\hat{K}_v$ together with a ϕ_e-equivariant simplicial map $\Phi_e : \hat{K}_e \to \hat{K}_v$

[24] no point is fixed by G

Theorem 3.13 ([33, special case of Section 3] See also [12]). *Let T be a real G-tree. Suppose that G is generic and that T is very small and super stable. Then, T has a model.*

Moreover, there is a model for T that is a graph of spaces such that each edge space is a point with non-trivial abelian stabilizer and each vertex space with restricted lamination is either

- (point) *a point with non-trivial stabilizer,*
- (linear) *a non-faithful action of an abelian group on the (2-skeleton of the) universal cover of a torus with an irrational* [25] *lamination, or*
- (surface) *a faithful action of a free group on the universal cover of a surface with non-empty boundary (represented by points with $\mathbb{Z}$-stabilizer) with a lamination of pure surface type.*

Remark 3.14. For an edge space $\{point\}$, the restriction of the lamination to $\{point\} \times I$ may or may not be empty. It can be checked that between any two points in models as in Theorem 3.13 there are Λ-length minimizing paths. Thin pieces do not arise because we are assuming our group is freely indecomposable.

Remark 3.15. Theorem 3.13 holds more generally if the assumption that G is freely indecomposable is replaced by the assumption that G is freely indecomposable rel point stabilizers, i.e. if $\mathcal{V}$ is the subset of G of elements acting elliptically[26] on T, then G cannot be expressed non-trivially as $A * B$ with all $g \in \mathcal{V}$ conjugate into $A \cup B$.

We can summarize Theorem 3.13 by saying that T is a non-trivial finite graph of simplicial trees, linear trees, and trees of pure surface type (over trivial trees). See Figure 2.

Corollary 3.16. *If G and T satisfy the hypotheses of Theorem 3.13, then G admits a non-trivial* GAD *Δ. Specifically, Δ may be taken to be the* GAD *induced by the boundary components of the surface vertex spaces and the simplicial edges of the model. The surface vertex spaces give rise to the* QH-*vertices of Δ and the linear vertex spaces give rise to the abelian vertices of Δ.*

[25] no essential loops in leaves
[26] fixing a point

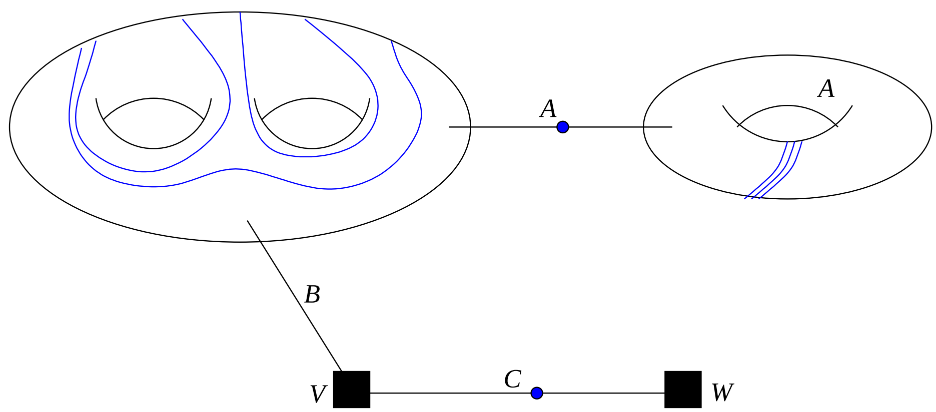

Figure 2: A model with a surface vertex space, a linear vertex space, and 2 rigid vertex spaces (the black boxes). The groups A, B and C are abelian with A and B infinite cyclic. Pieces of some leaves are also indicated by wavy lines and dots. For example, the dot on the edge labeled C is one leaf in a Cantor set of leaves.

3.4 Spaces of trees

Let G be a fg group and let $\mathcal{A}(G)$ be the set of minimal, non-trivial, real G-trees endowed with the Gromov topology[27]. Recall, see [22, 23, 4], that in the Gromov topology $\lim\{(T_n, d_n)\} = (T, d)$ if and only if: for any finite subset K of T, any $\epsilon > 0$, and any finite subset P of G, for sufficiently large n, there are subsets K_n of T_n and bijections $f_n : K_n \to K$ such that

$$|d(gf_n(s_n), f_n(t_n)) - d_n(gs_n, t_n)| < \epsilon$$

for all $s_n, t_n \in K_n$ and all $g \in P$. Intuitively, larger and larger pieces of the limit tree with their restricted actions appear in nearby trees.

Let $\mathcal{PA}(G)$ be the set of non-trivial real G-trees modulo homothety, i.e. $(T, d) \sim (T, \lambda d)$ for $\lambda > 0$. Fix a basis for $\mathbb{F}$ and let $T_{\mathbb{F}}$ be the corresponding Cayley graph. Give $T_{\mathbb{F}}$ the simplicial metric. So, a non-trivial homomorphism $f : G \to \mathbb{F}$ determines $T_f \in \mathcal{PA}(G)$. Let X be the subset of $Hom(G, \mathbb{F})$ consisting of those homomorphisms with non-cyclic image. The space of interest is the closure $\mathcal{T}(G)$ of (the image of) $\{T_f \mid f \in X\}$ in $\mathcal{PA}(G)$.

[27]The second author thanks Gilbert Levitt for a helpful discussion on the Gromov and length topologies.

Proposition 3.17 ([34]). *Every sequence of homomorphisms in X has a subsequence $\{f_n\}$ such that $\lim T_{f_n} = T$ in $\mathcal{T}(G)$. Further,*

1. *T is irreducible*[28].
2. *$\underrightarrow{Ker}\, f_n$ is precisely the kernel $Ker(T)$ of the action of G on T.*
3. *The action of $G/Ker(T)$ on T is very small and super stable.*
4. *For $g \in G$, $U(g) := \{T \in \mathcal{T}(G) \mid g \in Ker(T)\}$ is clopen*[29].

Proof. The initial statement follows from Paulin's Convergence Theorem [22].[30] The further items are exercises in Gromov convergence. ☐

Caution. Sela goes on to claim that stabilizers of minimal components of the limit tree are trivial (see Lemma 1.6 of [34]). However, it is possible to construct limit actions on the amalgam of a rank 2 free group F_2 and $\mathbb{Z}^3$ over $\mathbb{Z}$ where one of the generators of $\mathbb{Z}^3$ is glued to the commutator c of basis elements of F_2 and where the $\mathbb{Z}^3$ acts non-simplicially on a linear subtree with c acting trivially on the subtree but not in the kernel of the action. As a result, some of his arguments, though easily completed, are not fully complete.

Remark 3.18. There is another common topology on $\mathcal{A}(G)$, the length topology. For $T \in \mathcal{A}(G)$ and $g \in G$, let $\|g\|_T$ denote the minimum distance that g translates a point of T. The length topology is induced by the map $\mathcal{A}(G) \to [0,\infty)^G$, $T \mapsto (\|g\|_T)_{g\in G}$. Since the trees in $\mathcal{A}(G)$ are non-trivial, it follows from [42, page 64] that $\{0\}$ is not in the image[31]. Since $\mathcal{T}(G)$ consists of irreducible trees, it follows from [23] that the two topologies agree when restricted to $\mathcal{T}(G)$ and from [6] that $\mathcal{T}(G)$ injects into $\big([0,\infty)^G \setminus \{0\}\big)/(0,\infty)$.

Corollary 3.19. *$\mathcal{T}(G)$ is metrizable and compact.*

[28] T is not a line and doesn't have a fixed end

[29] both open and closed

[30] Paulin's proof assumes the existence of convex hulls and so does not apply in the generality stated in his theorem. His proof does however apply in our situation since convex hulls do exist in simplicial trees.

[31] In fact, if $\mathcal{B}_G$ is a finite generating set for G then $\|g\|_T \neq 0$ for some word g that is a product of at most two elements of $\mathcal{B}_G$.

Proof. $[0,\infty)^G \setminus \{0\} \to \big([0,\infty)^G \setminus \{0\}\big)/(0,\infty)$ has a section over $\mathcal{T}(G)$ (e.g. referring to Footnote 31, normalize so that the sum of the translation lengths of words in $\mathcal{B}_G$ of length at most two is one). Therefore, $\mathcal{T}(G)$ embeds in the metrizable space $[0,\infty)^G$. In light of this, the main statement of Proposition 3.17 implies that $\mathcal{T}(G)$ is compact. □

Remark 3.20. Culler and Morgan [6] show that, if G is fg, then $\mathcal{PA}(G)$ with the length topology is compact. This can be used instead of Paulin's convergence theorem to show that $\mathcal{T}(G)$ is compact. The main lemma to prove is that, in the length topology, the closure in $\mathcal{PA}(G)$ of $\{T_f \mid f \in X\}$ consists of irreducible trees.

4 Proof of the Main Proposition

To warm up, we first prove the Main Proposition under the additional assumption that Γ has only trivial abelian splittings, i.e. every simplicial Γ-tree with abelian edge stabilizers has a fixed point. This proof is then modified to apply to the general case.

Proposition 4.1. *Suppose that* Γ *is a generic limit group and has only trivial abelian splittings*[32]*. Then,* Γ *has a factor set.*

Proof. Let $T \in \mathcal{T}(\Gamma)$. By Proposition 3.17, either $Ker(T)$ is non-trivial or T satisfies the hypotheses of Theorem 3.13. The latter case doesn't occur or else, by Corollary 3.16, $\Gamma/Ker(T)$ admits a non-trivial abelian splitting. In particular, $Ker(T)$ is non-trivial. Choose non-trivial $k_T \in Ker(T)$. By Item 4 of Proposition 3.17, $\{U(k_T) \mid T \in \mathcal{T}(\Gamma)\}$ is an open cover of $\mathcal{T}(\Gamma)$. Let $\{U(k_i)\}$ be a finite subcover. By definition, $\{\Gamma \to Ab(\Gamma)\} \cup \{q_i : \Gamma \to \Gamma/\langle\langle k_i \rangle\rangle\}$ is a factor set. □

The key to the proof of the general case is Sela's notion of a *short* homomorphism, a concept which we now define.

Definition 4.2. Let G be an fg group. Two elements f and f' in $Hom(G,\mathbb{F})$ are *equivalent*, denoted $f \sim f'$, if there is $\alpha \in Mod(G)$ and an element $c \in \mathbb{F}$ such that $f' = i_c \circ f \circ \alpha$.[33] Fix a set $\mathcal{B}$ of generators for G and by $|f|$ denote

[32]By Proposition 3.17 and Corollary 3.16, generic limit groups have non-trivial abelian splittings. The purpose of this proposition is to illustrate the method in this simpler (vacuous) setting.

[33]i_c is conjugation by c

$\max_{g\in\mathcal{B}} |f(a)|$ where, for elements of $\mathbb{F}$, $|\cdot|$ indicates word length. We say that f is *short* if, for all $f' \sim f$, $|f| \leq |f'|$.

Note that if $f \in X$ and $f' \sim f$, then $f' \in X$. Here is another exercise in Gromov convergence. See [34, Claim 5.3] and also [28] and [2, Theorem 7.4].

Exercise 16. *Suppose that G is generic, $\{f_i\}$ is a sequence in $Hom(G, \mathbb{F})$, and $\lim T_{f_i} = T$ in $\mathcal{T}(G)$. Then, either*

- *$Ker(T)$ is non-trivial, or*
- *eventually f_i is not short.*

The idea is that if the first bullet does not hold, then the GAD of G given by Corollary 3.16 can be used to find elements of $Mod(G)$ that shorten f_i for i large.

Let Y be the subset of X consisting of short homomorphisms and let $\mathcal{T}'(G)$ be the closure in $\mathcal{T}(G)$ of $\{T_f \mid f \in Y\}$. By Corollary 3.19, $\mathcal{T}'(G)$ is compact.

Proof of the Main Proposition. Let $T \in \mathcal{T}'(\Gamma)$. By Exercise 16, $Ker(T)$ is non-trivial. Choose non-trivial $k_T \in Ker(T)$. By Corollary 3.19, $\{U(k_T) \mid T \in \mathcal{T}'(\Gamma)\}$ is an open cover of $\mathcal{T}'(\Gamma)$. Let $\{U(k_i)\}$ be a finite subcover. By definition, $\{\Gamma \to Ab(\Gamma)\} \cup \{q_i : \Gamma \to \Gamma/\langle\langle k_i\rangle\rangle\}$ is a factor set. □

Remark 4.3. Cornelius Reinfeldt and Richard Weidmann point out that a factor set for a generic limit group Γ can be found without appealing to Corollary 3.19 as follows. Let $\{\gamma_1, \dots\}$ enumerate the non-trivial elements of Γ. Let $\mathcal{Q}_i := \{\Gamma/\langle\langle\gamma_1\rangle\rangle, \dots, \Gamma/\langle\langle\gamma_i\rangle\rangle\}$. If $\mathcal{Q}_i$ is not a factor set, then there is $f_i \in Y$ that is injective on $\{\gamma_1, \dots, \gamma_i\}$. By Paulin's convergence theorem, a subsequence of $\{T_{f_i}\}$ converges to a faithful Γ-tree in contradiction to Exercise 16.

JSJ-decompositions will be used to prove Theorem 1.30, so we digress.

5 Review: JSJ-theory

Some familiarity with JSJ-theory is assumed. The reader is referred to Rips-Sela [29], Dunwoody-Sageev [7], Fujiwara-Papasoglou [8]. For any freely indecomposable fg group G consider the class GAD's with at most one edge such that:

(JSJ) every non-cyclic abelian subgroup $A \subset G$ is elliptic.

We observe that

- Any two such GAD's are hyperbolic-hyperbolic[34] or elliptic-elliptic[35] (a hyperbolic-elliptic pair implies that one splitting can be used to refine the other. Since the hyperbolic edge group is necessarily cyclic by (JSJ), this refinement gives a free product decomposition of G).
- A hyperbolic-hyperbolic pair has both edge groups cyclic and yields a GAD of G with a QH-vertex group.
- An elliptic-elliptic pair has a common refinement that satisfies (JSJ) and whose set of elliptics is the intersection of the sets of elliptics in the given splittings.

Given a GAD Δ of G, we say that $g \in G$ is Δ-*elliptic*[36] if there is a vertex group V of Δ such that either:

- V is QH and g is conjugate to a multiple of a boundary component;
- V is abelian and g is conjugate into $\overline{P}(V)$; or
- V is rigid and g is conjugate into the envelope of V.

The idea is that Δ gives rise to a family of splittings[37] with at most one edge that come from edges of the decomposition, from simple closed curves in QH-vertex groups, and from subgroups A' of an abelian vertex A that contain $\overline{P}(A)$ (equivalently $P(A)$) and with $A/A' \cong \mathbb{Z}$. For example, a non-peripheral element of A is hyperbolic in some 1-edge splitting obtained by blowing up the vertex A to an edge and then collapsing the original edges of Δ. An element is Δ-elliptic iff it is elliptic with respect to all these splittings with at most one edge. Conversely, any finite collection of GAD's with at most one edge and that satisfy (JSJ) gives rise to a GAD whose set of elliptics is precisely the intersection of the set of elliptics in the collection.

[34]each edge group of corresponding trees contains an element not fixing a point of the other tree

[35]each edge group of corresponding trees fixes a point of the other tree

[36]We thank Richard Weidmann for pointing out an error in a previous version of this definition and also for suggesting the correction.

[37]not necessarily satisfying (JSJ).

Definition 5.1. An *abelian* JSJ*-decomposition of* G is a GAD whose elliptic set is the intersection of elliptics in the family of *all* GAD's with at most one edge and that satisfy (JSJ).

Example 5.2. The group $G = F \times \mathbb{Z}$ has no 1-edge GAD's satisfying (JSJ) so the abelian JSJ-decomposition Δ of G is a single point labeled G. Of course, G does have (many) abelian splittings. If F is non-abelian, then every element of G is Δ-elliptic. If F is abelian, then only the torsion elements of G are Δ-elliptic.

To show that a group G admits an abelian JSJ-decomposition it is necessary to show that there is a bound to the complexity of the GAD's arising from finite collections of 1-edge splittings satisfying (JSJ). If G were fp the results of [3] would suffice. Since we don't know yet that limit groups are fp, another technique is needed. Following Sela, we use acylindrical accessibility.

Definition 5.3. A simplicial G-tree T is *n-acylindrical* if, for non-trivial $g \in G$, the diameter in the simplicial metric of the sets $Fix(g)$ is bounded by n. It is *acylindrical* if it is n-acylindrical for some n.

Theorem 5.4 (Acylindrical Accessibility: Sela [33], Weidmann [44]). *Let G be a non-cyclic freely indecomposable fg group and let T be a minimal k-acylindrical simplicial G-tree. Then, T/G has at most $1 + 2k(rank\ G - 1)$ vertices.*

The explicit bound in Theorem 5.4 is due to Richard Weidmann. For limit groups, 1-edge splittings satisfying (JSJ) are 2-acylindrical and finitely many such splittings give rise to GAD's that can be arranged to be 2-acylindrical. Theorem 5.4 can then be applied to show that abelian JSJ-decompositions exist.

Theorem 5.5 ([34]). *Limit groups admit abelian JSJ-decompositions.*

Exercise 17 (cf. Exercises 10 and 11). *If Γ is a generic limit group, then $Mod(\Gamma)$ is generated by inner automorphisms together with generalized Dehn twists associated to 1-edge splittings of Γ that satisfy (JSJ); see [34, Lemma 2.1]. In fact, the only generalized Dehn twists that are not Dehn twists can be taken to be with respect to a splitting of the form $A *_C B$ where $A = C \oplus \mathbb{Z}$.*

Remark 5.6. Suppose that Δ is an abelian JSJ-decomposition for a limit group G. If B is a rigid vertex group of Δ or the peripheral subgroup of an abelian vertex of Δ and if $\alpha \in Mod(G)$, then $\alpha|B$ is trivial[38]. Indeed, B is Δ'-elliptic in any 1-edge GAD Δ' of G satisfying (JSJ) and so the statement is true for a generating set of $Mod(G)$.

6 Limit groups are CLG's

In this section, we show that limit groups are CLG's and complete the proof of Theorem 1.30.

Lemma 6.1. *Limit groups are CLG's*

Proof. Let Γ be a limit group, which we may assume is generic. Let $\{f_i\}$ be a sequence in $Hom(\Gamma, \mathbb{F})$ such that f_i is injective on elements of length at most i (with respect to some finite generating set for Γ). Define $\hat{f}_i$ to be a short map equivalent to f_i. According to Exercise 16, $q : \Gamma \to \Gamma' := \Gamma/\underrightarrow{Ker}\ \hat{f}_i$ is a proper epimorphism, and so by induction we may assume that Γ' is a CLG.

Let Δ be an abelian JSJ-decomposition of Γ. We will show that q and Δ satisfy the conditions in Definition 1.25. The key observations are these.

- Elements of $Mod(\Gamma)$ when restricted to the peripheral subgroup $\overline{P}(A)$ of an abelian vertex A of Δ are trivial (Remark 5.6). Since $\underrightarrow{Ker}\ f_i$ is trivial, $q|\overline{P}(A)$ is injective. Similarly, the restriction of q to the envelope of a rigid vertex group of Δ is injective.

- Elements of $Mod(\Gamma)$ when restricted to edge groups of Δ are trivial. Since Γ is a limit group, each edge group is a maximal abelian subgroup in at least one of the two adjacent vertex groups. See Exercise 7.

- The q-image of a QH-vertex group Q of Δ is non-abelian. Indeed, suppose that Q is a QH-vertex group of Δ and that $q(Q)$ is abelian. Then, eventually $\hat{f}_i(Q)$ is abelian. QH-vertex groups of abelian JSJ-decompositions are canonical, and so every element of $Mod(\Gamma)$ preserves Q up to conjugacy. Hence, eventually $f_i(Q)$ is abelian, contradicting the triviality of $\underrightarrow{Ker}\ f_i$.

□

Proof of Theorem 1.30. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) were exercises. (3) $\Longrightarrow$ (1) is the content of Lemma 6.1. □

[38]Recall our convention that *trivial* means *agrees with the restriction of an inner automorphism.*

7 A more geometric approach

In this section, we show how to derive the Main Proposition using Rips theory for fp groups in place of the structure theory of actions of fg groups on real trees.

Definition 7.1. Let K be a finite 2-complex with a measured lamination (Λ, μ). The *length of* Λ, denoted $\|\Lambda\|$, is the sum $\Sigma_e \int_e \mu$ over the edges e of K.

If $\phi : \tilde{K} \to T$ is a resolution, then $\|\phi\|_K$ is the length of the induced lamination Λ_ϕ. Suppose that K is a 2-complex for G.[39] Recall that $T_{\mathbb{F}}$ is a Cayley graph for $\mathbb{F}$ with respect to a fixed basis and that from a homomorphism $f : G \to \mathbb{F}$ a resolution $\phi : (\tilde{K}, \tilde{K}^{(0)}) \to (T_{\mathbb{F}}, T_{\mathbb{F}}^{(0)})$ can be constructed, see [3]. The resolution ϕ depends on a choice of images of a set of orbit representatives of vertices in $\tilde{K}$. If ϕ minimizes $\| \cdot \|_K$ over this set of choices, then we define $\|f\|_K := \|\phi\|_K$.

Lemma 7.2. *Let K_1 and K_2 be finite 2-complexes for G. There is a number $B = B(K_1, K_2)$ such that, for all $f \in Hom(G, \mathbb{F})$,*

$$B^{-1} \cdot \|f\|_{K_1} \leq \|f\|_{K_2} \leq B \cdot \|f\|_{K_1}.$$

Proof. Let $\phi_1 : \tilde{K}_1 \to T_{\mathbb{F}}$ be a resolution such that $\|\phi_1\|_{K_1} = \|f\|_{K_1}$. Choose an equivariant map $\psi^{(0)} : \tilde{K}_2^{(0)} \to \tilde{K}_1^{(0)}$ between 0-skeleta. Then, $\phi_1 \psi^{(0)}$ determines a resolution $\phi_2 : \tilde{K}_2 \to T_{\mathbb{F}}$. Extend $\psi^{(0)}$ to a cellular map $\psi^{(1)} : \tilde{K}_2^{(1)} \to \tilde{K}_1^{(1)}$ between 1-skeleta. Let B_2 be the maximum over the edges e of the simplicial length of the path $\psi^{(1)}(e)$ and let E_2 be the number of edges in K_2. Then,

$$\|f\|_{K_2} \leq \|\phi_2\|_{K_2} \leq B_2 N_2 \|\phi_1\|_{K_1} = B_2 N_2 \|f\|_{K_1}.$$

The other inequality is similar. □

Recall that in Definition 4.2, we defined another length $| \cdot |$ for elements of $Hom(G, \mathbb{F})$.

Corollary 7.3. *Let K be a finite 2-complex for G. Then, there is a number $B = B(K)$ such that for all $f \in Hom(G, \mathbb{F})$*

$$B^{-1} \cdot |f| \leq \|f\|_K \leq B \cdot |f|.$$

[39] i.e. the fundamental group of K is identified with the group G

Proof. If $\mathcal{B}$ is the fixed finite generating set for G and if $R_{\mathcal{B}}$ is the wedge of circles with fundamental group identified with the free group on $\mathcal{B}$, then complete $R_{\mathcal{B}}$ to a 2-complex for G by adding finitely many 2-cells and apply Lemma 7.2. □

Remark 7.4. Lemma 7.2 and its corollary allow us to be somewhat cavalier with our choices of generating sets and 2-complexes.

Exercise 18. *The space of (nonempty) measured laminations on K can be identified with the closed cone without 0 in $\mathbb{R}^E_+$, where E is the set of edges of K, given by the triangle inequalities for each triangle of K. The projectivized space $\mathcal{PML}(K)$ is compact.*

Definition 7.5. Two sequences $\{m_i\}$ and $\{n_i\}$ in $\mathbb{N}$ are *comparable* if there is a number $C > 0$ such that $C^{-1} \cdot m_i \leq n_i \leq C \cdot m_i$ for all i.

Exercise 19. *Suppose K is a finite 2-complex for G, $\{f_i\}$ is a sequence in $Hom(G, \mathbb{F})$, $\phi_i : \tilde{K} \to T_{\mathbb{F}}$ is an f_i-equivariant resolution, $\lim T_{f_i} = T$, and $\lim \Lambda_{\phi_i} = \Lambda$. If $\{|f_i|\}$ and $\{\|\phi_i\|_K\}$ are comparable, then, there is a resolution $\tilde{K} \to T$ that sends lifts of leaves of Λ to points of T and is monotonic (Cantor function) on edges of $\tilde{K}$.*

Definition 7.6. An element f of $Hom(G, \mathbb{F})$ is *K-short* if $\|f\|_K \leq \|f'\|_K$ for all $f' \sim f$.

Corollary 7.7. *Let $\{f_i\}$ be a sequence in $Hom(G, \mathbb{F})$. Suppose that $f'_i \sim f_i \sim f''_i$ where f'_i is short and f''_i is K-short. Then, the sequences $\{|f'_i|\}$ and $\{\|f''_i\|_K\}$ are comparable.* □

Definition 7.8. If ℓ is a leaf of a measured lamination Λ on a finite 2-complex K, then (conjugacy classes of) elements in the image of $\pi_1(\ell \subset K)$ are *carried by ℓ*. Suppose that Λ_i is a component of Λ. If Λ_i is simplicial (consists of a parallel family of compact leaves ℓ), then elements in the image of $\pi_1(\ell \subset K)$ are *carried by* Λ_i. If Λ_i is minimal and if N is a standard neighborhood[40] of Λ_i, then elements in the image of $\pi_1(N \subset K)$ are *carried by* Λ_i.

Definition 7.9. Let K be a finite 2-complex for G. Let $\{f_i\}$ be a sequence of short elements in $Hom(G, \mathbb{F})$ and let $\phi_i : \tilde{K} \to T_{\mathbb{F}}$ be an f_i-equivariant resolution. We say that the sequence $\{\phi_i\}$ is *short* if $\{\|\phi_i\|_K\}$ and $\{|f_i|\}$ are comparable.

[40]see Theorem 3.6

Exercise 20. *Let G be freely indecomposable. In the setting of Definition 7.9, if $\{\phi_i\}$ is short, $\Lambda = \lim \Lambda_{\phi_i}$, and $T = \lim T_{f_i}$, then Λ has a leaf carrying non-trivial elements of $Ker(T)$.*

The idea is again that, if not, the induced GAD could be used to shorten. The next exercise, along the lines of Exercise 15, will be needed in the following lemma.[41]

Exercise 21. *Let Δ be a 1-edge GAD of a group G with a homomorphism q to a limit group Γ. Suppose:*

- *the vertex groups of Δ are non-abelian,*
- *the edge group of Δ is maximal abelian in each vertex group, and*
- *q is injective on vertex groups of Δ.*

Then, G is a limit group.

Lemma 7.10. *Let Γ be a limit group and let $q : G \to \Gamma$ be an epimorphism such that $Hom(G, \mathbb{F}) = Hom(\Gamma, \mathbb{F})$. If $\alpha \in Mod(G)$ then α induces an automorphism α' of Γ and α' is in $Mod(\Gamma)$.*

Proof. Since $\Gamma = RF(G)$, automorphisms of G induce automorphisms of Γ. Let Δ be a 1-edge splitting of G such that $\alpha \in Mod(\Delta)$. It is enough to check the lemma for α. We will check the case that $\Delta = A *_C B$ and that α is a Dehn twist by an element $c \in C$ and leave the other (similar) cases as exercises. We may assume that $q(A)$ and $q(B)$ are non-abelian for otherwise α' is trivial. Our goal is to successively modify q until it satisfies the conditions of Exercise 21.

First replace all edge and vertex groups by their q-images so that the third condition of the exercise holds. Always rename the result G. If the second condition does not hold, pull[42] the centralizers $Z_A(c)$ and $Z_B(c)$ across the edge. Iterate. It is not hard to show that the limiting GAD satisfies the conditions of the exercise. So, the modified G is a limit group. Since $Hom(G, \mathbb{F}) = Hom(\Gamma, \mathbb{F})$, we have that $G = \Gamma$ and $\alpha = \alpha'$. □

[41] It is a consequence of Theorem 1.30, but since we are giving an alternate proof we cannot use this.

[42] If A_0 is a subgroup of A, then the result of *pulling* A_0 across the edge is $A *_{\langle A_0, C\rangle} \langle A_0, B\rangle$, cf. moves of type IIA in [3].

Alternate proof of the Main Proposition. Suppose that Γ is a generic limit group, $T \in \mathcal{T}'(\Gamma)$, and $\{f_i\}$ is a sequence of short elements of $Hom(\Gamma, \mathbb{F})$ such that $\lim T_{f_i} = T$. As before, our goal is to show that $Ker(T)$ is non-trivial, so suppose it is trivial. Recall that the action of Γ on T satisfies all the conclusions of Proposition 3.17.

Let $q : G \to \Gamma$ be an epimorphism such that G is fp and $Hom(G, \mathbb{F}) = Hom(\Gamma, \mathbb{F})$. By Lemma 7.10, elements of the sequence $\{f_i q\}$ are short. We may assume that all intermediate quotients $G \to G' \to \Gamma$ are freely indecomposable[43].

Choose a 2-complex K for G and a subsequence so that $\Lambda = \lim \Lambda_{\phi_i}$ exists where $\phi_i : \tilde{K} \to T_{\mathbb{F}}$ is an $f_i q$-equivariant resolution and $\{\phi_i\}$ is short. For each component Λ_0 of Λ, perform one of the following moves to obtain a new finite laminated 2-complex for an fp quotient of G (that we will immediately rename (K, Λ) and G). Let G_0 denote the subgroup of G carried by Λ_0.

1. If Λ_0 is minimal and if G_0 stabilizes a linear subtree of T, then enlarge $N(\Lambda_0)$ to a model for the action of $q(G_0)$ on T.

2. If Λ_0 is minimal and if G_0 does not stabilize a linear subtree of T, then collapse all added annuli to their bases.

3. If Λ_0 is simplicial and G_0 stabilizes an arc of T, then attach 2-cells to leaves to replace G_0 by $q(G_0)$.

In each case, also modify the resolutions to obtain a short sequence on the new complex with induced laminations converging to Λ. The modified complex and resolutions contradict Exercise 20. Hence, $Ker(T)$ is non-trivial.

To finish, choose non-trivial $k_T \in Ker(T)$. As before, if $\{U(k_{T_i})\}$ is a finite cover for $\mathcal{T}'(\Gamma)$, then $\{\Gamma \to Ab(\Gamma)\} \cup \{\Gamma \to \Gamma/\langle\langle k_{T_i}\rangle\rangle\}$ is a factor set. $\square$

References

[1] Emina Alibegović. Makanin-Razborov diagrams for limit groups. *Geom. Topol.*, 11:643–666, 2007.

[43] see [30]

[2] Mladen Bestvina. $\mathbb{R}$-trees in topology, geometry, and group theory. In *Handbook of geometric topology*, pages 55–91. North-Holland, Amsterdam, 2002.

[3] Mladen Bestvina and Mark Feighn. Bounding the complexity of simplicial group actions on trees. *Invent. Math.*, 103(3):449–469, 1991.

[4] M. R. Bridson and G. A. Swarup. On Hausdorff-Gromov convergence and a theorem of Paulin. *Enseign. Math. (2)*, 40(3-4):267–289, 1994.

[5] Christophe Champetier and Vincent Guirardel. Limit groups as limits of free groups. *Israel J. Math.*, 146:1–75, 2005.

[6] Marc Culler and John W. Morgan. Group actions on **R**-trees. *Proc. London Math. Soc. (3)*, 55(3):571–604, 1987.

[7] M. J. Dunwoody and M. E. Sageev. JSJ-splittings for finitely presented groups over slender groups. *Invent. Math.*, 135(1):25–44, 1999.

[8] K. Fujiwara and P. Papasoglu. JSJ-decompositions of finitely presented groups and complexes of groups. *Geom. Funct. Anal.*, 16(1):70–125, 2006.

[9] R. I. Grigorchuk and P. F. Kurchanov. On quadratic equations in free groups. In *Proceedings of the International Conference on Algebra, Part 1 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 159–171, Providence, RI, 1992. Amer. Math. Soc.

[10] Daniel Groves. Limit groups for relatively hyperbolic group, I: the basic tools. preprint.

[11] Daniel Groves. Limit groups for relatively hyperbolic groups. II. Makanin-Razborov diagrams. *Geom. Topol.*, 9:2319–2358 (electronic), 2005.

[12] Vincent Guirardel. Actions of finitely generated groups on **R**-tree. to appear in Annales de l'Institut Fourier.

[13] Vincent Guirardel. Limit groups and groups acting freely on $\mathbb{R}^n$-trees. *Geom. Topol.*, 8:1427–1470 (electronic), 2004.

[14] O. Kharlampovich and A. Myasnikov. Irreducible affine varieties over a free group. I. Irreducibility of quadratic equations and Nullstellensatz. *J. Algebra*, 200(2):472–516, 1998.

[15] O. Kharlampovich and A. Myasnikov. Irreducible affine varieties over a free group. II. Systems in triangular quasi-quadratic form and description of residually free groups. *J. Algebra*, 200(2):517–570, 1998.

[16] O. Kharlampovich and A. Myasnikov. Description of fully residually free groups and irreducible affine varieties over a free group. In *Summer School in Group Theory in Banff, 1996*, volume 17 of *CRM Proc. Lecture Notes*, pages 71–80. Amer. Math. Soc., Providence, RI, 1999.

[17] Gilbert Levitt. La dynamique des pseudogroupes de rotations. *Invent. Math.*, 113(3):633–670, 1993.

[18] R. C. Lyndon. The equation $a^2b^2 = c^2$ in free groups. *Michigan Math. J*, 6:89–95, 1959.

[19] G. S. Makanin. Equations in a free group. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(6):1199–1273, 1344, 1982.

[20] G. S. Makanin. Decidability of the universal and positive theories of a free group. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(4):735–749, 1984.

[21] J. Morgan and P. Shalen. Valuations, trees, and degenerations of hyperbolic structures, I. *Ann. of Math.(2)*, 120:401–476, 1984.

[22] Frédéric Paulin. Topologie de Gromov équivariante, structures hyperboliques et arbres réels. *Invent. Math.*, 94(1):53–80, 1988.

[23] Frédéric Paulin. The Gromov topology on **R**-trees. *Topology Appl.*, 32(3):197–221, 1989.

[24] Frédéric Paulin. Sur la théorie élémentaire des groupes libres (d'après Sela). *Astérisque*, (294):ix, 363–402, 2004.

[25] A. A. Razborov. Systems of equations in a free group. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(4):779–832, 1984.

[26] V. N. Remeslennikov. ∃-free groups. *Sibirsk. Mat. Zh.*, 30(6):193–197, 1989.

[27] V. N. Remeslennikov. ∃-free groups and groups with length function. In *Second International Conference on Algebra (Barnaul, 1991)*, volume 184 of *Contemp. Math.*, pages 369–376. Amer. Math. Soc., Providence, RI, 1995.

[28] E. Rips and Z. Sela. Structure and rigidity in hyperbolic groups. I. *Geom. Funct. Anal.*, 4(3):337–371, 1994.

[29] E. Rips and Z. Sela. Cyclic splittings of finitely presented groups and the canonical JSJ decomposition. *Ann. of Math. (2)*, 146(1):53–109, 1997.

[30] G. P. Scott. Finitely generated 3-manifold groups are finitely presented. *J. London Math. Soc. (2)*, 6:437–440, 1973.

[31] Z. Sela. Diophantine geometry over groups VII: The elementary theory of a hyperbolic group. preprint.

[32] Z. Sela. Diophantine geometry over groups VIII: Stability. preprint.

[33] Z. Sela. Acylindrical accessibility for groups. *Invent. Math.*, 129(3):527–565, 1997.

[34] Z. Sela. Diophantine geometry over groups I. Makanin-Razborov diagrams. *Publ. Math. Inst. Hautes Études Sci.*, (93):31–105, 2001.

[35] Z. Sela. Diophantine geometry over groups and the elementary theory of free and hyperbolic groups. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 87–92, Beijing, 2002. Higher Ed. Press.

[36] Z. Sela. Diophantine geometry over groups II. Completions, closures and formal solutions. *Israel J. Math.*, 134:173–254, 2003.

[37] Z. Sela. Diophantine geometry over groups IV. An iterative procedure for validation of a sentence. *Israel J. Math.*, 143:1–130, 2004.

[38] Z. Sela. Diophantine geometry over groups III. Rigid and solid solutions. *Israel J. Math.*, 147:1–73, 2005.

[39] Z. Sela. Diophantine geometry over groups V_1. Quantifier elimination. I. *Israel J. Math.*, 150:1–197, 2005.

[40] Z. Sela. Diophantine geometry over groups V_2. Quantifier elimination. II. *Geom. Funct. Anal.*, 16(3):537–706, 2006.

[41] Z. Sela. Diophantine geometry over groups VI. The elementary theory of a free group. *Geom. Funct. Anal.*, 16(3):707–730, 2006.

[42] Jean-Pierre Serre. *Trees*. Springer-Verlag, Berlin, 1980. Translated from the French by John Stillwell.

[43] John Stallings. How not to prove the Poincaré conjecture. In *Topology Seminar, Wisconsin, 1965*, Edited by R. H. Bing and R. J. Bean. Annals of Mathematics Studies, No. 60, pages ix+246. Princeton University Press, 1966.

[44] Richard Weidmann. The Nielsen method for groups acting on trees. *Proc. London Math. Soc. (3)*, 85(1):93–118, 2002.

[45] Henry Wilton. Solutions to Bestvina and Feighn's Exercises on limit groups. math.GR/0604137.

[46] Heiner Zieschang. Alternierende Produkte in freien Gruppen. *Abh. Math. Sem. Univ. Hamburg*, 27:13–31, 1964.