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CLASSIFYING THE ACTIONS OF FINITE GROUPS ON ORIENTABLE SURFACES OF GENUS 4

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Abstract

We work out a classification of all effective orientation-preserving finite group actions, up to topological equivalence, on an orientable closed surface of genus 4.

Key words and phrases: orientable closed surface, orientation-preserving finite group actions.

Classification of finite groups of homeomorphisms of surfaces is a classical problem which has been attracting a considerable interest of mathematicians since the last century and which still remains unsolved. It is of great importance for the module problem and for studying of the structure of mapping class groups. An extensive literature devoted to this question is cited in [2, 13].

Let T_σ be an orientable closed surface of genus σ. The symbol $\mathrm{Homeom}(T_\sigma)$ stands for the group of all homeomorphisms of the surface, $\mathrm{Isot}(T_\sigma)$ is the subgroup consisting of homeomorphisms isotopic to identity, and

$$M_\sigma = \mathrm{Homeom}(T_\sigma)/\mathrm{Isot}(T_\sigma)$$

is the mapping class group of T_σ. The group M_σ is isomorphic to the group $\mathrm{Out}\,\pi_1(T_\sigma)$ of outer automorphisms of the fundamental group of the surface T_σ [10].

If we confine exposition to orientation-preserving homeomorphisms then we can define the subgroups $\mathrm{Homeom}^+(T_\sigma)$ and M_σ^+ of index 2 in the groups $\mathrm{Homeom}(T_\sigma)$ and M_σ, respectively. For $\sigma \geq 2$, there exists a one-to-one correspondence between the conjugacy classes of finite subgroups in the groups $\mathrm{Homeom}^+(T_\sigma)$ and M_σ^+ (see [2: p. 233]).

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Finite subgroups in M_σ^+ for $\sigma = 2, 3, 4, 5$ are classified up to isomorphism in [12, 8, 7, 6], respectively. In the last two articles the technique of actions on the space of holomorphic differentials is used. However, it was noted in [2: p. 237] that this technique does not allow us to classify orientation-preserving actions of finite groups on T_σ up to topological equivalence.

In the present article, we classify all effective orientation-preserving actions of finite groups, up to topological equivalence, on the surface T_4. This classification together with the Reidemeister–Schreier method makes it possible to easily enumerate representatives of the conjugacy classes of finite subgroups in M_4^+.

We use the technique of [2] where similar results were obtained for the surfaces T_2 and T_3.

In Section 1, we describe our approach to the study of finite group actions on surfaces, state the main theorem, and correct an error of [2]. In Section 2 auxiliary results are presented and Section 3 is devoted to the proof of the main theorem.

1. Algebraic approach to the study of G-actions. The main theorem

In what follows, we use the notation $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$, where x and y are elements of a group. Let T_σ be an orientable closed surface of genus $\sigma \geq 2$ and let $\mathrm{Homeom}^+(T_\sigma)$ be the group of its orientation-preserving homeomorphisms. Given a finite group G, by a G-action we mean a monomorphism $\varepsilon\colon G \to \mathrm{Homeom}^+(T_\sigma)$. Two G-actions ε and ε' are said to be *topologically equivalent* if there exist an automorphism $\alpha \in \mathrm{Aut}(G)$ and a homeomorphism $h \in \mathrm{Homeom}^+(T_\sigma)$ such that $\varepsilon'(g) = h^{-1}\varepsilon(\alpha(g))h$ for every $g \in G$.

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$ be the hyperbolic plane. A discontinuous group of orientation-preserving motions of this plane is called a *Fuchsian group*. Assume given a Fuchsian group S isomorphic to the fundamental group $\pi_1(T_\sigma)$. Then the space of orbits, $\mathbb{H}/S$, is homeomorphic to T_σ. Identify it with T_σ. Let M be the normalizer of S in the group $\mathrm{Homeom}^+(\mathbb{H})$. Each homeomorphism of the plane $\mathbb{H}$ in M induces a homeomorphism of the surface T_σ which belongs to $\mathrm{Homeom}^+(T_\sigma)$. The corresponding map $f\colon M \to \mathrm{Homeom}^+(T_\sigma)$ is an epimorphism with kernel S.

Let $\varepsilon\colon G \to \mathrm{Homeom}^+(T_\sigma)$ be some G-action. In view of [13: Section 7.3], a genetic code of the group $f^{-1}(\varepsilon(G))$ is of the form

$$G^* = \left\langle \alpha_1, \beta_1, \dots, \alpha_\rho, \beta_\rho,\ \gamma_1, \dots, \gamma_r \;\middle|\; \prod_{i=1}^{\rho}[\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = 1,\ \gamma_1^{m_1} = \dots = \gamma_r^{m_r} = 1 \right\rangle. \qquad (1)$$

The numbers $\rho, m_1, \ldots, m_r$ are uniquely determined if we assume $2 \leq m_1 \leq \cdots \leq m_r$. This follows, for example, from the proof of Lemma 4.7.4 in [13]. We say that $(\rho : m_1, \ldots, m_r)$ are the branching data for the G-action ε. Equivalent G-actions have equal branching data. On the other hand, as we show below, nonequivalent G-actions can have equal branching data too.

Recall the following fundamental facts.

1. The branching data $(\rho : m_1, \ldots, m_r)$ are connected with σ and $|G|$ by the Riemann–Hurwitz formula

$$(2\sigma - 2)/|G| = 2\rho - 2 + \sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right). \tag{2}$$

From here we can easily derive the following Hurwitz theorem: $|G| \leq 84(\sigma - 1)$ for $\sigma \geq 2$; given $\sigma \geq 2$ and $|G|$, there exists at most finitely many sets $(\rho : m_1, \ldots, m_r)$ satisfying (2). Throughout the sequel, we consider integer-valued sets for which $\rho \geq 0$ and $2 \leq m_1 \leq \cdots \leq m_r$.

2. For a G-action with the branching data $(\rho : m_1, \ldots, m_r)$, there exists an epimorphism $\eta \colon G^* \to G$, where G^* is defined by (1), whose kernel is torsion-free (it is isomorphic to $\pi_1(T_\sigma)$).

Let x_i, y_i, and z_j be the images of α_i, β_i, and γ_j under the action of the epimorphism $\eta \colon G^* \to G$. It is known that $|\gamma_j| = m_j$ and, since $\operatorname{Ker}\eta$ is torsion-free, $|z_j| = m_j$. Hence, the group G is generated by the elements x_i, y_i, and z_j $(1 \leq i \leq \rho, 1 \leq j \leq r)$ such that

$$\prod_{i=1}^{\rho}[x_i, y_i]\prod_{j=1}^{r} z_j = 1, \tag{3}$$

$$|z_j| = m_j \quad (j = 1, \ldots, r). \tag{4}$$

In particular, we have

$$m_j \mid |G| \quad (j = 1, \ldots, r). \tag{5}$$

Definition 1. An ordered set $(x_1, \ldots, x_\rho, y_1, \ldots, y_\rho; z_1, \ldots, z_r)$ of elements in G is called a *generating* $(\rho : m_1, \ldots, m_r)$*-vector* if G is generated by these elements and, in addition, formulas (3) and (4) hold.

Below we define an equivalence relation on the set of generating $(\rho : m_1, \ldots, m_r)$-vectors of G.

Let $\widehat{G^*}$ be a free group with free generators $a_1, \ldots, a_\rho, b_1, \ldots, b_\rho, c_1, \ldots, c_r$. There is a natural epimorphism $\widehat{G^*} \to G^*$ transforming a_i, b_i $(1 \leq i \leq \rho)$, and c_j $(1 \leq j \leq r)$ into α_i, β_i, and γ_j, respectively. Denote the word $\prod_{i=1}^{\rho}[a_i, b_i]\prod_{j=1}^{r} c_j$ by w. In accord with [13: Theorem 5.8.2], each automorphism $\alpha \in \operatorname{Aut}(G^*)$ is induced by an automorphism $\widehat{\alpha} \in \operatorname{Aut}(\widehat{G^*})$ such that $\widehat{\alpha}(w)$ is conjugate to w^δ in $\widehat{G^*}$ and $\widehat{\alpha}(c_j)$ is conjugate to $c_{\pi(j)}^{\varepsilon_j}$ $(1 \le j \le r)$. Here $\delta, \varepsilon_j \in \{-1, 1\}$ and π is a permutation on the symbols $1, \dots, r$ such that $|\gamma_{\pi(j)}| = |\gamma_j|$ for $1 \le j \le r$. In this case, δ is independent of the choice of $\widehat{\alpha}$.

Let $\mathrm{Aut}^+(G^*)$ consist of automorphisms $\alpha \in \mathrm{Aut}(G^*)$ such that $\delta = 1$. Then $\mathrm{Aut}^+(G^*)$ is a subgroup of index 2 in $\mathrm{Aut}(G^*)$. Define an action of $\mathrm{Aut}^+(G^*) \times \mathrm{Aut}(G)$ on the set of generating $(\rho : m_1, \dots, m_r)$-vectors of G.

Let $(\alpha, \beta) \in \mathrm{Aut}^+(G^*) \times \mathrm{Aut}(G)$ and let $u = (x_1, \dots, x_\rho, y_1, \dots, y_\rho; z_1, \dots, z_r)$ be a generating $(\rho : m_1, \dots, m_r)$-vector for G. Put $u(\alpha, \beta) = u'$ whenever the vector u' is obtained from the vector $(\alpha_1, \dots, \alpha_\rho, \beta_1, \dots, \beta_\rho; \gamma_1, \dots, \gamma_r)$ by applying the map $\alpha\eta\beta$ to each of its entries, where $\eta\colon G^* \to G$ is an epimorphism transforming α_i, β_i, and γ_j $(i = 1, \dots, \rho;\ j = 1, \dots, r)$ into x_i, y_i, and z_j, respectively.

Generating $(\rho : m_1, \dots, m_r)$-vectors v and v' of G are referred to be *equivalent* if $v(\alpha, \beta) = v'$ for some pair $(\alpha, \beta) \in \mathrm{Aut}^+(G^*) \times \mathrm{Aut}(G)$.

Proposition 1 (see [2, 5, 9]). *Let formula* (2) *hold for* σ, $|G|$, *and* $(\rho : m_1, \dots, m_r)$. *Then there exists a bijection between the set of equivalence classes of G-actions on T_σ with the branching data $(\rho : m_1, \dots, m_r)$ and the set of equivalence classes of generating $(\rho : m_1, \dots, m_r)$-vectors of G.*

Before stating the main theorem, we make the following remarks.

Remark 1. If numbers $n_1, \dots, n_s$ occur in the branching data with multiplicities $k_1, \dots, k_s$ then we write $(\rho : n_1^{k_1}, \dots, n_s^{k_s})$ rather than

$$\Bigl(\rho : \underbrace{n_1, \dots, n_1}_{k_1}, \dots, \underbrace{n_s, \dots, n_s}_{k_s}\Bigr).$$

In what follows we omit ρ in the branching data if $\rho = 0$.

Remark 2. Below we use the following genetic codes:
$\langle x \mid x^n = 1 \rangle$ for the cyclic group $\mathbb{Z}_n$,
$\langle x, y \mid x^n = y^m = [x, y] = 1 \rangle$ for the group $\mathbb{Z}_n \times \mathbb{Z}_m$,
$\langle x, y \mid x^2 = 1,\ y^n = 1,\ x^{-1}yx = y^{-1} \rangle$ for the dihedral group D_n.
The symbol $D_{n,m,k}$ denotes the group with genetic code

$$\langle x, y \mid x^n = 1,\ y^m = 1,\ x^{-1}yx = y^k \rangle.$$

Remark 3. A quasidihedral group of order 2^n is a group isomorphic to $D_{2,2^{n-1},2^{n-2}-1}$ $(n \ge 4)$.

Throughout the sequel, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group of order 8; S_n and A_n are the symmetric and alternating groups on n symbols; $GL_n(q)$ and $SL_n(q)$ are the general and special linear groups of matrices of order $n \times n$ over a field of q elements.

Theorem. *A finite group admits an effective orientation-preserving action on an orientable closed surface of genus 4 if and only if it is imbeddable into one of the following nine groups:*

$$\mathbb{Z}_{15},\ \ \mathbb{Z}_{18},\ \ SL_2(3),\ \ S_3\times\mathbb{Z}_6,\ \ S_4\times\mathbb{Z}_3,\ \ S_5,$$
$$\text{a quasidihedral group of order 32},\ \ \mathbb{Z}_5 \leftthreetimes D_4,\ \ (\mathbb{Z}_3\times\mathbb{Z}_3)\leftthreetimes D_4,$$

where the dihedral group D_4 of order 8 acts on $\mathbb{Z}_3\times\mathbb{Z}_3$ without kernel and on $\mathbb{Z}_5$ with a noncyclic kernel of order 4.

The equivalence classes of G-actions, where G is a nonidentity finite group, are in one-to-one correspondence with the triples (G, the branching data, the generating vector) listed in Tables 1 and 2.

The scheme of the proof is as follows. First, we enumerate the sets $(|G|,(\rho : m_1,\dots,m_r))$ that satisfy (5) and (2) for $\sigma = 4$ and are such that all the numbers m_j occur in the list (9). Next, for such a set, we find groups of order $|G|$ with generating $(\rho : m_1,\dots,m_r)$-vectors and find representatives of equivalence classes of these vectors. For abelian and dihedral groups, in case $\rho = 0$, the representatives are listed in Table 1 without any explanations, since they are easily obtained either directly or with the help of Subsections 2.3 and 2.4. The remaining cases are examined in Section 3.

Tables 1 and 2 are composed with regard to the above remarks. The elements $A, B, C \in GL_2(3) \cong \mathrm{Aut}(\mathbb{Z}_3\times\mathbb{Z}_3)$ are specified in the following lemma.

Lemma 1. *The group $GL_2(3)$ contains exactly two conjugacy classes of involutions with representatives $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, one conjugacy class of elements of order 4 with a representative $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, one conjugacy class of subgroups isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$ with a representative $\langle A, B\rangle$, and one conjugacy class of subgroups isomorphic to D_4 with a representative $\langle C, B\rangle$.*

Remark. In the tables below, A, B, and C are the automorphisms of the group $\mathbb{Z}_3\times\mathbb{Z}_3$ given by the following equalities:

$$A(x) = x^{-1},\ \ A(y) = y^{-1};$$
$$B(x) = x^{-1},\ \ B(y) = y;$$
$$C(x) = y^{-1},\ \ C(y) = x.$$

Table 1

Groups	Branching data	Generating vectors	Groups	Branching data	Generating vectors
$\mathbb{Z}_2$	(2^{10})	$\big(\underbrace{x, x, \dots, x}_{10}\big)$	$\mathbb{Z}_8$	$(2^2, 8^2)$	(x^4, x^4, x, x^{-1})
	$(1 : 2^6)$	$\big(1, 1; \underbrace{x, x, \dots, x}_{6}\big)$	D_4	$(2^4, 4)$	(x, x, x, xy^{-1}, y)
	$(2 : 2^2)$	$(1, 1, 1, 1; x, x)$			$(x, xy^{-1}, y^2, y^2, y)$
$\mathbb{Z}_3$	(3^6)	$\big(\underbrace{x, x, \dots, x}_{6}\big)$	$\mathbb{Z}_9$	(9^3)	(x, x, x^{-2})
		(x, x, x, x^2, x^2, x^2)	$\mathbb{Z}_3 \times \mathbb{Z}_3$	(3^4)	(x, y, x, xy^2)
	$(1 : 3^3)$	$(1, 1; x, x, x)$			(x, y, x^2, y^2)
	$(2 : -)$	$(x, 1, 1, 1)$	$\mathbb{Z}_{10}$	$(5, 10^2)$	(x^2, x, x^{-3})
$\mathbb{Z}_4$	$(1 : 4^2)$	$(1, 1; x, x^{-1})$			(x^{-2}, x, x)
	$(2^4, 4^2)$	$(x^2, x^2, x^2, x^2, x, x^{-1})$		$(2^2, 5^2)$	(x^5, x^5, x^2, x^{-2})
	$(2, 4^4)$	(x^2, x, x, x, x^{-1})	D_5	$(2^2, 5^2)$	(x, x, y, y^{-1})
$\mathbb{Z}_2 \times \mathbb{Z}_2$	(2^7)	(x, x, x, y, y, y, xy)			(x, xy^2, y, y^2)
		$\big(\underbrace{x, x, \dots, x}_{5}, y, xy\big)$	$\mathbb{Z}_{12}$	$(3, 12^2)$	(x^4, x^7, x)
	$(1 : 2^3)$	$(1, 1; x, y, xy)$		$(4, 6, 12)$	(x^{-3}, x^2, x)
$\mathbb{Z}_5$	(5^4)	(x, x^2, x^3, x^4)	D_6	(2^5)	(x, x, xy, xy^4, y^3)
		(x, x, x^{-1}, x^{-1})		$(2^2, 3, 6)$	(x, xy, y^{-2}, y)
		(x, x, x, x^2)	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$(2^2, 3, 6)$	(x, y^3, y^2, yx)
$\mathbb{Z}_6$	$(2^3, 3, 6)$	(x^3, x^3, x^3, x^2, x)		(6^3)	(y, xy, xy^{-2})
	$(3^2, 6^2)$	(x^2, x^2, x, x)	$\mathbb{Z}_{15}$	$(3, 5, 15)$	(x^5, x^{-6}, x)
		(x^2, x^{-2}, x^{-1}, x)	$\mathbb{Z}_{16}$	$(2, 16^2)$	(x^8, x, x^7)
	$(2, 6^3)$	(x^3, x, x, x)	D_8	$(2^3, 8)$	(x, xy^3, y^4, y)
	$(2^2, 3^3)$	$(x^3, x^3, x^2, x^2, x^2)$	$\mathbb{Z}_{18}$	$(2, 9, 18)$	(x^9, x^8, x)
	$(1 : 2^2)$	$(x, 1; x^3, x^3)$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$(3, 6^2)$	$(x, x^{-1}y^{-1}, y)$
D_3	$(2^2, 3^3)$	(x, x, y, y, y)	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	$(2, 10^2)$	$(x, y, y^{-1}x)$
	$(1 : 2^2)$	$(y, 1; x, x)$	D_{10}	$(2^3, 5)$	(x, xy^3, y^5, y^2)
	(2^6)	(x, x, x, x, x^y, x^y)			

Table 2

Groups	Branching data	Generating vectors
Q_8	$(2,4^3)$	$(-1,i,j,k)$
$A_4 = \langle x,y \mid x=(12)(34),\ y=(123)\rangle$	$(2,3^3)$	$(x,y,y,y^{-2}x)$
	$(1:2)$	$(x,y;[x,y]^{-1})$
$\langle x,y \mid y^8=1,\ x^2=y^4,\ x^{-1}yx=y^{-1}\rangle$	$(4^2,8)$	$(x,x^{-1}y^{-1},y)$
$(\mathbb{Z}_3\times\mathbb{Z}_3)\leftthreetimes\langle B\rangle$	$(3,6^2)$	(xy,yB,xyB)
		$(x,yB,xy^{-1}B)$
	$(2^2,3^2)$	(B,Bx^{-1},y^{-1},xy)
		(B,Bx,xy^{-1},xy)
$(\mathbb{Z}_3\times\mathbb{Z}_3)\leftthreetimes\langle A\rangle$	$(2^2,3^2)$	$(A,Ay^{-1}x^{-1},x,y)$
$D_{4,5,-1}$	$(4^2,5)$	$(x,x^{-1}y^{-1},y)$
$D_{4,5,2}$	$(4^2,5)$	$(x,x^{-1}y^{-1},y)$
S_4	$(2^3,4)$	$((12),(13),(14),(4321))$
$Q_8\leftthreetimes\langle y\mid y^3=1\rangle\cong SL_2(3)$	$(3,4,6)$	$(y,-i,iy^{-1})$
$D_{2,12,-5}$	$(2,6,12)$	(x,xy^{-1},y)
$D_{2,16,7}$	$(2,4,16)$	(x,xy^{-1},y)
$(\mathbb{Z}_3\times\mathbb{Z}_3)\leftthreetimes\langle A,B\rangle$	$(2^3,3)$	$(A,BAy,Bx,x^{-1}y^{-1})$
	$(2,6^2)$	(Axy^{-1},By,BAx)
$S_3\times\langle u\mid u^6=1\rangle$	$(2,6^2)$	$((12),(123)u,(13)u^{-1})$
$(\mathbb{Z}_3\times\mathbb{Z}_3)\leftthreetimes\langle C\rangle$	$(3,4^2)$	$(y,C^{-1}x,C)$
$A_4\times\langle x\mid x^3=1\rangle$	$(3^2,6)$	$((123),(143)x^{-1},(12)(34)x)$
$\langle x,y,z\mid x^5=y^4=z^2=1,\ y^z=y^{-1},$ $x^z=x,\ x^y=x^{-1}\rangle\cong\mathbb{Z}_5\leftthreetimes D_4$	$(2,4,10)$	(xzy,y,y^2zx^{-1})
A_5	$(2,5^2)$	$((24)(35),(12345),(13452))$
$S_4\times\langle x\mid x^3=1\rangle$	$(2,3,12)$	$((14),(123)x,(1324)x^{-1})$
$(\mathbb{Z}_3\times\mathbb{Z}_3)\leftthreetimes\langle B,C\rangle$	$(2,4,6)$	(Bx,C,BCx)
S_5	$(2,4,5)$	$((12),(2543),(12345))$

Finally, we mention an error in [2: see p. 263–264 and Table 5]. It is asserted there that there exist two nonequivalent actions of the group

$$G=\langle x,y,z \mid x^2=y^2=z^4=[y,z]=[x,z]=1,\ xyx^{-1}=yz^2\rangle\cong(\mathbb{Z}_4\times\mathbb{Z}_2)\leftthreetimes\mathbb{Z}_2$$

on the surface T_3 with branching data $(2^3,4)$. The former is defined by the generating vector (x,xzy,y,z^{-1}) and the latter by the vector (x,xzy,z^2,yz) (in Table 5 the last vector was written incorrectly). In fact, the last vector is not generating since the group $\langle x,xzy,z^2,yz\rangle=\langle x,yz\rangle$ has the normal subgroup $\langle yz\rangle$ of index 2 and order 4 and, thus, is not equal to G.

It may be proven that a G-action on T_3 with branching data $(2^3,4)$ is unique up to topological equivalence.

2. Auxiliary results

2.1. *The groups* $\mathrm{Aut}^+(G^*)$ *for* $\rho = 0, 1$. By convention, we write out an automorphism action on a generator if and only if this generator is not fixed.

Proposition 2 (see [1,2]). *Let* G^* *and* $\widehat{G^*}$ *be the above-introduced groups, where* ρ *is equal to* 0 *or* 1. *For* $\rho = 0$, *we consider the following automorphisms of the group* $\mathrm{Aut}(\widehat{G^*})$:

$$\theta_j\colon c_j \to c_{j+1}, \quad c_{j+1} \to c_{j+1}^{-1} c_j c_{j+1} \quad (1 \le j \le r-1).$$

For $\rho = 1$ *we additionally consider the automorphisms*

$$\varphi\colon a_1 \to b_1 a_1, \quad \psi\colon b_1 \to a_1 b_1,$$

$$\mu_j : \begin{cases} a_1 \to a_1 c_r^{-1} \cdots c_{j+1}^{-1} c_{j-1}^{-1} \cdots c_1^{-1} b_1^{-1}, \\ c_j \to t c_j t^{-1}, \text{ where } t = c_{j+1} \cdots c_r b_1 c_1 \cdots c_{j-1}, \end{cases} \qquad j \in \{1, \ldots, r\}.$$
$$\nu_j : \begin{cases} b_1 \to b_1 a_1 c_r^{-1} \cdots c_{j+1}^{-1} c_{j-1}^{-1} \cdots c_1^{-1}, \\ c_j \to t c_j t^{-1}, \text{ where } t = c_{j+1} \cdots c_r a_1^{-1} c_1 \cdots c_{j-1}, \end{cases}$$

The finite product $\widehat{p}$ *of the above automorphisms, their inverses, and inner automorphisms induces an automorphism belonging to the group* $\mathrm{Aut}^+(G^*)$ *if and only if the images of* $\widehat{p}(c_j)$ *and* c_j *in* G^* *are of the same order* $(1 \le j \le r)$. *For* $\rho = 0$, *every automorphism* $p \in \mathrm{Aut}^+(G^*)$ *is induced by such an automorphism* $\widehat{p} \in \mathrm{Aut}\big(\widehat{G^*}\big)$.

2.2. *The Singerman method of intermediate actions.* The following proposition is immediate from [11].

Proposition 3. *Let* $\varepsilon\colon G \to \mathrm{Homeom}^+(T_\sigma)$ *be an action with branching data* $(\rho : m_1, \ldots, m_r)$ *and let* $(x_1, \ldots, x_\rho,\ y_1, \ldots, y_\rho;\ z_1, \ldots, z_r)$ *be the corresponding generating* $(\rho : m_1, \ldots, m_r)$*-vector for* G. *Assume that* $H \le G$ *and the branching data for the action* $\varepsilon\big|_H\colon H \to \mathrm{Homeom}^+(T_\sigma)$ *are* $(\tau : n_1, \ldots, n_t)$. *Consider the action of the group* G *on the set of the right cosets of* G/H *by the right multiplication. Let an element* z_i *have* k_i *orbits of length* $m_i/l_{i1}, \ldots, m_i/l_{ik_i}$. *Then the set* $(n_1, \ldots, n_t)$ *is obtained by deleting unities in* $(l_{11}, \ldots, l_{1k_1}, \ldots, l_{r1}, \ldots, l_{rk_r})$ *and* τ *is determined by the equality*

$$|G : H| \left(2\rho - 2 + \sum_{j=1}^{r} \left(1 - \frac{1}{m_j} \right) \right) = \left(2\tau - 2 + \sum_{s=1}^{t} \left(1 - \frac{1}{n_s} \right) \right).$$

It often suffices to know that

$$\text{every } n_s \text{ is a divisor of some } m_j. \tag{6}$$

Proposition 4. *Let the conditions of Proposition 3 be satisfied for $\tau = 0$ (in this case, $\rho = 0$). Then the subgroup H is normal in G if and only if, for each $i = 1, \ldots, r$, the lengths of all orbits under the action $\langle z_i \rangle$ on G/H by the right multiplication coincide.*

It was noted in [2] that this proposition follows readily from [3].

We will use Proposition 4 in the situation in which the branching data for the G-action ε and possible branching data for H-actions are known, but the generating vectors are unknown.

Example. Describe the classes of actions for groups of order 24 on T_4 with branching data $(3, 4, 6)$.

Assume that $|G| = 24$, $\varepsilon\colon G \to \mathrm{Homeom}^+(T_4)$ is an action with branching data $(3, 4, 6)$, and (x_1, x_2, x_3) is a generating $(3, 4, 6)$-vector for G corresponding to ε.

In view of formulas (2) and (5), applied to an arbitrary group H of order 8, actions of H on T_4 may have the following branching data: $(1 : 4)$, $(2^4, 4)$, $(2, 4^3)$, and $(2^2, 8^2)$. Let H be a Sylow 2-subgroup in G and let G/H be the set of right cosets of G by H. To determine the branching data for the action $\varepsilon|_H$, we need to study actions of the subgroups $\langle x_i \rangle$ on G/H by the right multiplication.

The subgroups $\langle x_1 \rangle$ and $\langle x_3 \rangle$ act transitively on G/H, since otherwise one-element orbits arise and the number 3 or 6 appears in the branching data for the action $\varepsilon|_H$. Therefore, $\langle x_1 \rangle$ gives nothing for the branching data of $\varepsilon|_H$ and $\langle x_3 \rangle$ gives the number 2. The group $\langle x_2 \rangle$ has either three orbits of length 1 or one orbit of length 1 and one orbit of length 2; the group $\langle x_2 \rangle$ brings in the branching data for $\varepsilon|_H$ three 4's in the former case and 4 and 2 in the latter case. We see that only the former case holds. Therefore, the branching data for H are $(2, 4^3)$, and $H \trianglelefteq G$.

Since the product of three elements of order 4 in abelian or dihedral groups of order 8 is not an element of order 2, we infer $H \cong Q_8$.

Let $L = \langle y \rangle$ be a Sylow 3-subgroup in G. Observe that it is not unique. Otherwise, $\langle x_1, x_2, x_3 \rangle = \langle x_1, (x_1^{-1} x_3^{-1}), x_3 \rangle = \langle x_1, x_3 \rangle = \langle x_3 \rangle \neq G$. Therefore, y acts on H by conjugation, as an element of order 3 in $\mathrm{Aut}(H)$. Since all elements of order 3 in $\mathrm{Aut}(Q_8)$ are conjugated, we infer

$$G = \langle Q_8, y \mid y^3 = 1,\ y^{-1} i y = j,\ y^{-1} j y = k,\ y^{-1} k y = i \rangle \cong SL_2(3).$$

We now prove that each generating $(3, 4, 6)$-vector (x_1, x_2, x_3) for G is equivalent to the vector $(y, -i, iy^{-1})$ relative to the action of the group $\mathrm{Aut}^+(G^*) \times \mathrm{Aut}(G)$. Applying an appropriate automorphism in $\mathrm{Aut}(G)$, we may assume that $x_1 = y$. The group $\langle y \rangle$, acting by conjugation on the set of all elements of order 4, gives two orbits, $\{i, j, k\}$ and $\{-i, -j, -k\}$. Therefore, we may assume that either $x_2 = i$ or $x_2 = -i$. The former does not suit us, since in this case $|x_3| = |x_1 x_2| = 3$. Thus, we arrive at the vector $(y, -i, iy^{-1})$.

So, there exists at most one class of actions of groups of order 24 on T_4 with branching data $(3, 4, 6)$.

In what follows, for an arbitrary action $\varepsilon \colon G \to \mathrm{Homeom}^+(T_4)$ and a subgroup $H < G$, we write out possible branching data for the action $\varepsilon\big|_H$, omitting a detailed analysis of actions of the generating vector entries on the set of right cosets G/H. If this is the case, we make a choice among the already known branching data for H-actions.

2.3. *Actions of cyclic groups.* Let G be a finite (not necessarily cyclic) group and let $\varepsilon \colon G \to \mathrm{Homeom}^+(T_\sigma)$ be an action with branching data $(\rho : m_1, \dots, m_r)$ whose generating $(\rho : m_1, \dots, m_r)$-vector is $(x_1, \dots, x_\rho, y_1, \dots, y_\rho; z_1, \dots, z_r)$. Given a nonunity element $g \in G$, the symbol $|T_\sigma^g|$ stands for the number of fixed points of g on T_σ.

Recall (see [2, 4]) that

$$|T_\sigma^g| = \big|N_G(\langle g \rangle)\big| \sum_{j=1}^{r} \frac{\delta_j(g)}{m_j}, \tag{7}$$

where $\delta_j(g)$ equals 1 if g is conjugate to a power of z_j and 0 otherwise;

if $G = \langle g \rangle$ then $|T_\sigma^g|$ is equal to the number of j's for which $|G| = m_j$. (8)

Proposition 5 [5]. *Let the group $\mathbb{Z}_n$ act on the surface T_σ of genus $\sigma \geq 2$ with branching data $(\rho : m_1, \dots, m_r)$. Then every equivalence class of generating $(\rho : m_1, \dots, m_r)$-vectors contains the vector $(x_1, \dots, x_\rho, y_1, \dots, y_\rho; z_1, \dots, z_r)$, where $x_2 = \dots = x_\rho = y_1 = \dots = y_\rho = 0$ and x_1 generates $\mathbb{Z}_n/\mathbb{Z}_m$, $m = \mathrm{l.c.m.}(m_1, \dots, m_r)$. Furthermore, with the use of the transforms $\mu_1, \dots, \mu_r$, we can make sure that the vectors $(x_1, 1, \dots, 1; z_1, \dots, z_r)$ and $(x_1', 1, \dots, 1; z_1, \dots, z_r)$ are equivalent whenever $x_1' x_1^{-1} \in \langle z_1, \dots, z_r \rangle$.*

Proposition 6 [2: p. 256]. *Let the group $\mathbb{Z}_n$ $(n > 1)$ act on the surface T_σ of genus $\sigma \geq 2$. Then*

1) $n \leq 4\sigma + 2$;
2) *if n is a prime number then either $n = 2\sigma + 1$ or $n \leq \sigma + 1$;*
3) $\varphi(n) \leq 2\sigma$, *where φ is the Euler function.*

This proposition yields that possible values of n for $\sigma = 4$ are

$$2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 15,\ 16,\ 18. \tag{9}$$

Our theorem implies that these values are realized.

2.4. *Actions of dihedral type groups for $\rho = 0$.* In this subsection, A denotes a finite abelian group nonisomorphic to $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and the symbol $\widetilde{A}$ stands for the extension of dihedral type:

$$\widetilde{A} = \langle A, x \mid x^2 = 1,\ a^x = a^{-1}\ (a \in A) \rangle.$$

Hence, A is an automorphically admissible subgroup of $\widetilde{A}$.

We say that ordered sets $(a_1', \ldots, a_s')$ and $(a_1'', \ldots, a_s'')$ of elements in A are *equivalent* if we can pass from one to the other using a finite number of the following transformations:

n_{ij}: interchanges a_i and a_j provided that $|a_i| = |a_j|$, $i \neq j$;

n_i: takes the inverse of a_i;

τ: substitutes $\tau(a_k)$ for a_k, $k = 1, \ldots, s$ $(\tau \in \mathrm{Aut}(A);\ i, j \in \{1, \ldots, s\})$.

An ordered set $(a_1, \ldots, a_s)$ is called a *weakly generating* $(k_1, \ldots, k_s)$*-vector* for A if $2 \leq k_1 \leq \cdots \leq k_s$, $|a_1| = k_1$, $\ldots$, $|a_s| = k_s$, and $\langle a_1, \ldots, a_s \rangle = A$ (cf. Definition 1).

Given a generating vector $(z_1, \ldots, z_r)$ for the group $\widetilde{A}$, its *A-part* is the vector obtained by deleting the entries which does not lie in A.

Lemma 2. *Let $(z_1', \ldots, z_r')$ and $(z_1'', \ldots, z_r'')$ be equivalent generating $(m_1, \ldots, m_r)$-vectors for $\widetilde{A}$ (see Section 1). Then their A-parts are equivalent (in the above sense).*

The *proof* follows from Proposition 2.

Lemma 3. *Any generating $(m_1, \ldots, m_r)$-vector with $m_4 \geq 3$ for the group $\widetilde{A}$ is equivalent to a generating vector of the form $(x, xa_r^{-1} \cdots a_3^{-1}, a_3, \ldots, a_r)$, where $(a_3, \ldots, a_r)$ is a weakly generating $(m_3, \ldots, m_r)$-vector for A.*

Proof. Let $(z_1, \ldots, z_r)$ be a generating $(m_1, \ldots, m_r)$-vector for $\widetilde{A}$. Since $3 \leq m_4 \leq \cdots \leq m_r$, we have $z_4, \ldots, z_r \in A$. Using the relations $\langle z_1, \ldots, z_r \rangle = \widetilde{A}$ and $z_1 z_2 \cdots z_r = 1$, we conclude that exactly two elements of (z_1, z_2, z_3) lie in the coset xA and the third one, together with $z_4, \ldots, z_r$, generates A. Applying one of the transformations id, θ_2, and $\theta_1\theta_2$, we can obtain a vector $(z_1', z_2', z_3', z_4, \ldots, z_r)$ such that $z_1', z_2' \in xA$ and $z_3' \in A$.

Let $z_1' = xa$, $a \in A$. Then the automorphism

$$\tau\colon \begin{cases} x \to xa^{-1}, \\ b \to b \quad (b \in A) \end{cases}$$

transforms the vector $(z_1', z_2', z_3', z_4, \ldots, z_r)$ into the vector $(x, z'', z_3', z_4, \ldots, z_r)$, where $z'' = xz_r^{-1} \cdots z_4^{-1}(z_3')^{-1}$, since the product of all entries is equal to 1. It remains to assign $a_3 = z_3'$ and $a_i = z_i$ for $i \geq 4$.

Lemma 4. *Let $m_1 \leq \cdots \leq m_r$, $m_1 = m_2 = 2$, and $m_4 \geq 3$. Then there exists a bijection between the equivalence classes of generating $(m_1, \ldots, m_r)$-vectors for $\widetilde{A}$ and the equivalence classes of weakly generating $(m_3, \ldots, m_r)$-vectors for A.*

Proof. Define a map Ψ from the set of weakly generating $(m_3, \ldots, m_r)$-vectors for A into the set of generating $(2, 2, m_3, \ldots, m_r)$-vectors for $\widetilde{A}$ as follows: the image of $(a_3, \ldots, a_r)$ is the vector $\left(x, xa_r^{-1} \cdots a_3^{-1}, a_3, \ldots, a_r\right)$. Prove that Ψ agrees with the equivalence relation. It suffices to show that the vectors $t_b = \left(x, xb_r^{-1} \cdots b_3^{-1}, b_3, \ldots, b_r\right)$ and $t_a = \left(x, xa_r^{-1} \cdots a_3^{-1}, a_3, \ldots, a_r\right)$ are equivalent whenever $(b_3, \ldots, b_r)$ is the image of $(a_3, \ldots, a_r)$ under one of the maps n_{ij}, n_i, and τ.

To obtain the vector t_b from the vector t_a, in the cases n_{ij} $(1 \leq i < j \leq r-2)$, n_i, and $\tau \in \operatorname{Aut}(A)$ we use the respective transformations $\theta_{i+2} \cdots \theta_j \theta_{j+1} \theta_j^{-1} \cdots \theta_{i+2}^{-1}$, $\theta_{i+1}\theta_i \cdots \theta_3 \theta_2^2 \theta_3^{-1} \cdots \theta_i^{-1} \theta_{i+1}^{-1}$, and the automorphism $\widehat{\tau} \in \operatorname{Aut}(\widetilde{A})$, $\widehat{\tau}(x) = x$ and $\widehat{\tau}(y) = \tau(y)$ for $y \in A$.

Therefore, we may define a natural map Ψ^* from the set of equivalence classes of weakly generating $(m_3, \ldots, m_r)$-vectors for A into the set of equivalence classes of generating $(2, 2, m_3, \ldots, m_r)$-vectors for $\widetilde{A}$. By Lemma 3, Ψ^* is surjective and, by Lemma 2, Ψ^* is one-to-one.

Lemma 5. *For the group $D_3 = \langle x, y \mid x^2 = y^3 = 1,\ x^{-1}yx = y^{-1}\rangle$, any generating (2^6)-vector is equivalent to the vector (x, x, x, x, x^y, x^y).*

Proof. Let $(x_1, \ldots, x_6)$ be a generating (2^6)-vector for D_3. Denote by $n(x)$, $n(x^y)$, and $n\left(x^{y^2}\right)$ the number of its entries equal to x, x^y, and x^{y^2}. The group D_3 acts by conjugation on the set $\left\{x, x^y, x^{y^2}\right\}$ of all its involutions as the whole permutation group on three symbols. Conjugating the vector $(x_1, \ldots, x_6)$ by a suitable element of D_3, we may suppose that

$$n(x) \geq n(x^y) \geq n\left(x^{y^2}\right), \quad n(x) + n(x^y) + n\left(x^{y^2}\right) = 6. \tag{10}$$

Next, applying appropriate transformations of the form θ_j, we may assume that all x's are situated before x^y and the latter in turn are situated before x^{y^2}. In the set of all these vectors, the only vectors that satisfy (10) and the properties $x_1 \cdots x_6 = 1$ and $\langle x_1, \ldots, x_6\rangle = D_3$ are

$$\left(x, x, x^y, x^y, x^{y^2}, x^{y^2}\right) \text{ and } (x, x, x, x, x^y, x^y).$$

But they are equivalent, since the second vector is obtained from the first one with the help of the transformation $\theta_4^{-1}\theta_3\theta_5^{-1}\theta_4$.

Lemma 6. *For the group $D_4 = \langle x, y \mid x^2 = y^4 = 1,\ x^{-1}yx = y^{-1}\rangle$, there exist only two equivalence classes of generating $(2^4, 4)$-vectors; they are defined by the vectors (x, x, x, xy^{-1}, y) and $(x, xy^{-1}, y^2, y^2, y)$.*

Proof. Let $(x_1, \dots, x_5)$ be a generating $(2^4, 4)$-vector for D_4. Conjugating it by the element x if necessary, we may assume $x_5 = y$. Two cases are possible.

1. Assume that the involutions x_1, x_2, x_3, x_4 are not central. In this case, some of them coincide (otherwise they are equal to the involutions x, xy, xy^2, and xy^3 up to permutation and $x_1 \cdots x_5 \neq 1$ even modulo the commutator subgroup). Using transformations of the form $\theta_j^{\pm 1}$, we may assume that $x_1 = x_2$ and, using an automorphism of the form $x \to xy^i$, $y \to y$, we may assume that $x_1 = x_2 = x$. Then $x_3 x_4 = x_2^{-1} x_1^{-1} x_5^{-1} = y^{-1}$ and we have the following four possibilities for the pair (x_3, x_4): $(x_3, x_4) = (xy^i, xy^{i-1})$, $i = 0, 1, 2, 3$. Since the group $\langle \theta_3 \rangle$ acts transitively on the set of vectors $(x, x, xy^i, xy^{i-1}, y)$, $i = 0, 1, 2, 3$, all the vectors are equivalent to the vector (x, x, x, xy^{-1}, y).

2. Assume that some of the elements x_1, x_2, x_3, and x_4 are equal to the central involution y^2. In view of the condition $x_1 x_2 \cdots x_5 = 1$, there are only two such elements. Using transformations of the form $\theta_j^{\pm 1}$, we may assume that $x_3 = x_4 = y^2$. Then $x_1 x_2 = x_5^{-1} x_4^{-1} x_3^{-1} = y^{-1}$ and, arguing as at the end of case 1, we arrive at the vector $(x, xy^{-1}, y^2, y^2, y)$.

The vectors (x, x, x, xy^{-1}, y) and $(x, xy^{-1}, y^2, y^2, y)$ are nonequivalent since transformations of the form θ_j and automorphisms preserve the number of vector entries lying in the center.

Lemma 7. *For the group $D_6 = \langle x, y \mid x^2 = y^6 = 1,\ x^{-1}yx = y^{-1}\rangle$, any generating (2^5)-vector is equivalent to the vector (x, x, xy, xy^4, y^3).*

Proof. Let $(x_1, \dots, x_5)$ be a generating (2^5)-vector for D_6. Since all its entries are involutions and $x_1 \cdots x_5 = 1$, one of the entries is equal to y^3. Applying a transformation of the form θ_j^{-1}, we may assume $x_5 = y^3$. Prove that $x_i \neq y^3$ for $i = 1, \dots, 4$. Assume the contrary. Since $x_1 \cdots x_5 = 1$, at least two elements among the first four are equal to y^3. As before, we may assume that $x_3 = x_4 = y^3$. Then $x_1 x_2 = y^3$ and $\langle x_1, \dots, x_5\rangle = \langle x_1, y^3\rangle \neq G$, which yields a contradiction.

Therefore, the elements x_i $(i = 1, \dots, 4)$ are noncentral involutions and, thus, they are of the form xy^{k_i}, $k_i \in \{0, 1, 2, 3, 4, 5\}$. We prove that there exists a vector equivalent to $(x_1, \dots, x_4, y^3)$ and such that two entries among the first four coincide.

Assume that all k_i are distinct. In this case, there exist subscripts i and j such that $1 \leq i < j \leq 4$ and $|k_i - k_j| = 1$. After the transformation $\theta_{j-1}\theta_{j-2} \cdots \theta_{i+1}$ the elements xy^{k_i} and xy^{k_j} will occupy ith and $(i+1)$th positions. Observe that after the subsequent application of the transformation

θ_i^s we obtain the elements $xy^{k_i+s(k_j-k_i)}$ and $xy^{k_j+s(k_j-k_i)}$ on these positions. Let the element xy^{k_t} occupy another position (different from i and $i+1$). Then this element occupies the position i for $s=(k_t-k_i)/(k_j-k_i)$.

Thus, we may assume that $x_i = x_j$ for some i and j $(1 \le i < j \le 4)$. With the help of transformations of the form θ_j and automorphisms of the form $\nu_t\colon x \to xy^t$, $y \to y$, we can obtain $x_1 = x_2 = x$. Then $x_3x_4 = x_2^{-1}x_1^{-1}x_5^{-1} = y^3$. Hence, $(x_3, x_4) = (xy^a, xy^{a+3})$, $a \in \{0, \pm1, \pm2, 3\}$. Taking it into account that $\langle x_1, \dots, x_5\rangle = D_6$, we obtain $a \in \{\pm1, \pm2\}$. The automorphism $x \to x$, $y \to y^{-1}$ substitutes $-a$ for a. The transformation θ_3 substitute $a+3$ (mod 6) for a. Therefore, each generating (2^5)-vector for D_6 is equivalent to (x, x, xy, xy^4, y^3).

3. The proof of the main theorem

3.1. *Elimination of some groups.*

Lemma 8. *Groups of orders* 27, 30, 45, 48, 54, 90, *and* 108 *cannot act on T_4 effectively while preserving the orientation.*

Proof. Let G be one of the above-mentioned groups. Assume that the group G acts on T_4 effectively and preserves orientation. Then only $\rho = 0$ and (m_1, m_2, m_3) listed below satisfy (2), (5), and the condition that all m_i are in the list (9). Let (x_1, x_2, x_3) be a generating (m_1, m_2, m_3)-vector for G. In each of the cases below we arrive at a contradiction.

1. Let $|G| = 27$ and let the branching data be $(3^2, 9)$.

Since G is nilpotent, we have $\langle x_3\rangle \trianglelefteq G$. This and the equality $G = \langle x_2, x_3\rangle$ imply $G = \langle x_3\rangle \leftthreetimes \langle x_2\rangle$. Let $x_2^{-1}x_3x_2 = x_3^t$. Then $1 = x_1^3 = \left(x_3^{-1}x_2^{-1}\right)^3 = x_3^{-1}\cdot x_2^{-1}x_3^{-1}x_2\cdot x_2^{-2}x_3^{-1}x_2^2 = x_3^{-t^2-t-1}$. However, the congruence $t^2+t+1 \equiv 0 \pmod 9$ is not solvable, a contradiction.

Using this fact and the Sylow theorem, we eliminate the cases $|G| = 54$ (the branching data are $(2,3,18)$) and $|G| = 108$ (the branching data are $(2,3,9)$).

2. Let $|G| = 30$ and let the branching data be $(2,5,10)$.

An arbitrary group of order 10 can only have the branching data $(2^2, 5^2)$ or $(5, 10^2)$. However, this contradicts the results of Subsection 2.2 applied to G and $H = \langle x_3\rangle$.

3. Let $|G| = 45$ and let the branching data be $(3^2, 5)$.

This case can be excluded, since x_1 and x_2 lie in the unique Sylow 3-subgroup and do not generate G.

4. Let $|G| = 48$ and let the branching data be $(2,4,8)$.

Let H be a Sylow 2-subgroup in G. The following branching data are possible for an arbitrary group of order 16: $(2^3, 8)$, $(4^2, 8)$, and $(2, 16^2)$. None of them is appropriate for H in view of Subsection 2.2.

5. Let $|G| = 90$ and let the branching data be $(2, 3, 10)$.
The group G is not simple, since the order of any finite simple noncyclic group is divisible by 4. Therefore, the group G has a normal subgroup of index 2, 3, or 5. We can exclude the first two possibilities taking into account cases 2 and 3 already considered. Assume now that $H \trianglelefteq G$ and $|G : H| = 5$. Let $(\tau : n_1, \ldots, n_t)$ be the branching data for H. The Riemann–Hurwitz formula for H implies that $\tau = 0$ and the numbers $n_1, \ldots, n_t$ do not contain five twos. However, this fact contradicts Proposition 4.

3.2. *Actions with branching data* $(\rho : m_1, \ldots, m_r)$, *where* $\rho > 0$. All the orders $|G|$ and the sets $(\rho : m_1, \ldots, m_r)$, $\rho > 0$, satisfying (2) for $\sigma = 4$, condition (5), and such that the numbers $m_1, \ldots, m_r$ are in the list (9) are presented below. Some of the cases do not lead to G-actions.

1. $|G| = 2$, $(1 : 2^6)$.
2. $|G| = 2$, $(2 : 2^2)$.
3. $|G| = 3$, $(2 : -)$.

In these cases, the equivalence classes of G-actions are unique and determined with the help of Proposition 5 (see Table 1).

4. $|G| = 3$, $(1 : 3^3)$.

Let $G = \langle x \mid x^3 = 1 \rangle$. By Proposition 5 we can choose a generating vector $(\alpha_1, \beta_1; \gamma_1, \gamma_2, \gamma_3)$ to satisfy the equality $\alpha_1 = \beta_1 = 1$. Next, since γ_1, γ_2, and γ_3 are elements of order 3 and $1 = [\alpha_1, \beta_1]\gamma_1\gamma_2\gamma_3 = \gamma_1\gamma_2\gamma_3$, all γ_i are equal to either x or x^2. The last case can be reduced to the former if we use an automorphism of the group G. Thus, each generating $(1 : 3^3)$-vector for G is equivalent to $(1, 1; x, x, x)$.

5. $|G| = 4$, $(1 : 2^3)$.

Let $(\alpha_1, \beta_1; \gamma_1, \gamma_2, \gamma_3)$ be a generating $(1 : 2^3)$-vector for G. It is easily seen that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $G = \big\langle x, y \bigm| x^2 = y^2 = 1,\ xy = yx \big\rangle$. The equality $1 = [\alpha_1, \beta_1]\gamma_1\gamma_2\gamma_3 = \gamma_1\gamma_2\gamma_3$ implies that the involutions γ_1, γ_2, and γ_3 are distinct. Employing $\mathrm{Aut}(G)$, we may assume that $\gamma_1 = x$, $\gamma_2 = y$, and $\gamma_3 = xy$. In this case, the transformations $\mu_i\varphi$ and $\nu_j\psi^{-1}$ are as follows: $\mu_i\varphi\colon \alpha_1 \to \alpha_1\gamma_i$; $\nu_j\psi^{-1}\colon \beta_1 \to \beta_1\gamma_j$. Using them, we obtain $\alpha_1 = \beta_1 = 1$. Thus, every generating $(1 : 2^3)$-vector for G is equivalent to the vector $(1, 1; x, y, xy)$.

6. $|G| = 4$, $(1 : 4^2)$.

We have $G \cong \mathbb{Z}_4$. Let $(\alpha_1, \beta_1; \gamma_1, \gamma_2)$ be a generating $(1 : 4^2)$-vector for $G = \langle x \mid x^4 = 1 \rangle$. The equality $[\alpha_1, \beta_1]\gamma_1\gamma_2 = 1$ implies that one of the elements γ_1 and γ_2 is equal to x and the other to x^{-1}. Considering the transformation θ_1, we may assume that $\gamma_1 = x$ and $\gamma_2 = x^{-1}$. By Proposition 5, we derive that the initial vector is equivalent to $(1, 1; x, x^{-1})$.

7. $|G| = 6$, $(1 : 2^2)$.

The following two cases are possible: $G \cong \mathbb{Z}_6$ and $G \cong S_3$.

(a) $G = \langle x \mid x^6 = 1 \rangle \cong \mathbb{Z}_6$.

Proposition 5 implies that every generating $(1:2^2)$-vector for G is equivalent either to $(x, 1; x^3, x^3)$ or to $(x^{-1}, 1; x^3, x^3)$. But these vectors are $\mathrm{Aut}(G)$-equivalent.

(b) $G = \langle x, y \mid x^2 = y^3 = 1,\ x^{-1}yx = y^{-1}\rangle \cong S_3$.
Let $(\alpha_1, \beta_1; \gamma_1, \gamma_2)$ be a generating $(1:2^2)$-vector. We show that it is equivalent to the vector $(y, 1; x, x)$. Assume the involutions γ_1 and γ_2 to be distinct. Since $[\alpha_1, \beta_1]\gamma_1\gamma_2 = 1$, either α_1 or β_1 is an involution. Without loss of generality, we may assume that β_1 is an involution, since the transformation $\varphi\psi^{-1}$ takes $(\alpha_1, \beta_1; \gamma_1, \gamma_2)$ into $(\beta_1\alpha_1, \alpha_1^{-1}; \gamma_1, \gamma_2)$. Since θ_1^{-1} transforms $(\alpha_1, \beta_1; \gamma_1, \gamma_2)$ into $(\alpha_1, \beta_1; \gamma_1\gamma_2\gamma_1^{-1}, \gamma_1)$ and $\gamma_1 \neq \gamma_2$, we may assume that $\beta_1 \neq \gamma_2$. Then the element $\gamma_2\beta_1$ is of order 3 and presents a product of different involutions.

We now consider the chain of equivalent vectors

$$(\alpha_1, \beta_1; \gamma_1, \gamma_2) \stackrel{\mu_1}{\to} (*, \beta_1; (\gamma_2\beta_1)\gamma_1(\gamma_2\beta_1)^{-1}, \gamma_2) \stackrel{\mu_1}{\to} (*, \beta_1; (\gamma_2\beta_1)^2\gamma_1(\gamma_2\beta_1)^{-2}, \gamma_2).$$

Since the triple cycle acts transitively on the set of involutions, the element $(\gamma_2\beta_1)^i\gamma_1(\gamma_2\beta_1)^{-i}$ coincides with γ_2 for some i. Therefore, in what follows we assume $\gamma_1 = \gamma_2$. The condition $[\alpha_1, \beta_1]\gamma_1\gamma_2 = 1$ implies that $[\alpha_1, \beta_1] = 1$, i.e., α_1 is a power of β_1 or vice versa. With the help of transformations φ and ψ, we may obtain $\beta_1 = 1$.

Since $(\alpha_1, 1; \gamma_1, \gamma_1)$ is a generating vector and $|\gamma_1| = 2$, either α_1 or $\alpha_1\gamma_1$ is a triple cycle. Without loss of generality, we assume $|\alpha_1| = 3$, since

$$(\alpha_1, 1; \gamma_1, \gamma_2) \stackrel{\mu_1}{\to} (\alpha_1\gamma_1, 1; \gamma_1, \gamma_1).$$

It remains to note that the group $\mathrm{Aut}(G)$ acts transitively on the set of all pairs (a triple cycle, an involution). Therefore, the vector $(\alpha_1, 1; \gamma_1, \gamma_1)$ is equivalent to the vector $(y, 1; x, x)$.

8. $|G| = 8$, $(1:4)$.
In this case, a G-action is impossible, since the commutator subgroup of any group of order 8 has order at most 2 and, thus, condition (3) fails.

9. $|G| = 9$, $(1:3)$.
This case cannot be realized too, since any group of order 9 is abelian and we arrive at a contradiction with condition (3).

10. $|G| = 12$, $(1:2)$.
Let $(\alpha_1, \beta_1; \gamma_1)$ be a generating $(1:2)$-vector for G; in particular, $[\alpha_1, \beta_1]\gamma_1 = 1$ and $|\gamma_1| = 2$. Therefore, G is a nonabelian group, $G \not\cong D_{4,3,-1}$, and $G \not\cong D_6$ (in the last two cases, $G' \cong \mathbb{Z}_3$). It remains only to consider the case

$$G \cong A_4 = \langle x, y \mid x = (12)(34),\ y = (123)\rangle.$$

Prove first that, for arbitrary triple cycles d_1 and d_2 in A_4, either d_2d_1 or $d_2^{-1}d_1$ belongs to the Klein group K. Indeed, the group K acts transitively by conjugation on four Sylow 3-subgroups of the group G. Therefore, there exists a $k \in K$ such that either $d_2 = k^{-1}d_1k$ or $d_2 = k^{-1}d_1^{-1}k$. In the former case, $d_2^{-1}d_1 = k^{-1}d_1^{-1}kd_1 \in G' = K$ and, in the latter, $d_2d_1 = k^{-1}d_1^{-1}kd_1 \in K$. Therefore, we may assume that either α_1 or β_1 lie in K (otherwise, applying the transformation φ or φ^{-1}, we can replace α_1 by $\beta_1\alpha_1$ or $\beta_1^{-1}\alpha_1$ and refer to the above arguments). The elements α_1 and β_1 are not in K simultaneously; otherwise $\langle\alpha_1, \beta_1, \gamma_1\rangle \neq G$. Thus, either $\alpha_1 \in K$ and $\beta_1 \notin K$ or $\alpha_1 \notin K$ and $\beta_1 \in K$.

Since $(\alpha_1, \beta_1; \gamma_1) \overset{\psi\varphi^\varepsilon}{\to} ((\alpha_1\beta_1)^\varepsilon\alpha_1, \alpha_1\beta_1; \gamma_1)$, the last case can be reduced to the first one in view of the above assertion for an appropriate $\varepsilon = \pm 1$. Therefore, assume $|\alpha_1| = 2$ and $|\beta_1| = 3$. But $\mathrm{Aut}(A_4)$ acts transitively on the set of pairs whose first element is of order 2 and the second is of order 3. Consequently, the vector $(\alpha_1, \beta_1; \gamma_1)$ is equivalent to the vector $(x, y; [x, y]^{-1})$.

3.3. *Actions with the branching data* $(0 : m_1, \dots, m_r)$. Below we examine possible orders of groups G and possible branching data for $\rho = 0$. In the case of abelian or dihedral groups, we shall not specify in detail how the generating vectors in Table 1 can be obtained. For the abelian case, it can be done directly. For the dihedral case, generating vectors can be found with the use of Lemmas 3–7. In particular, we begin with the case $|G| = 6$.

Remark. If the number $n = |G|$ occurs in the branching data for G then $G \cong \mathbb{Z}_n$.

1. $|G| = 6$.

In this case, the following branching data are possible: $(2^3, 3, 6)$, $(3^2, 6^2)$, $(2, 6^3)$, (2^6), and $(2^2, 3^3)$. For the first three variants, $G \cong \mathbb{Z}_6$; for the forth, $G \cong D_3$, since $\mathbb{Z}_6$ is not generated by involutions; for the fifth, either $G \cong \mathbb{Z}_6$ or $G \cong D_3$.

2. $|G| = 8$.

We use the fact that a group of order 8 is either abelian or isomorphic to D_4 or Q_8.

2.1. $(2^2, 8^2)$. Then $G \cong \mathbb{Z}_8$.

2.2. $(2^4, 4)$. Then G is nonabelian, since, in an abelian group, the product of elements of order 2 is not an element of order 4. Next, $G \not\cong Q_8$, since Q_8 contains only one element of order 2 and, hence, condition (3) does not hold. Thus, $G \cong D_4$.

2.3. $(2, 4^3)$. Then G is nonabelian and $G \not\cong D_4$, since in D_4 the elements of order 4 generate a proper subgroup. It remains to analyze the group Q_8.

Let (x_1, x_2, x_3, x_4) be a generating $(2, 4^3)$-vector for Q_8. Then $x_1 = -1$ and $x_2, x_3, x_4 \in \{\pm i, \pm j, \pm k\}$. Moreover, the elements $x_2^{\pm 1}$, $x_3^{\pm 1}$, and $x_4^{\pm 1}$ are distinct; otherwise condition (3) is violated not only in Q_8 but also in the quotient group $Q_8/\langle x_1\rangle$. Since the group $\mathrm{Aut}(Q_8)$ acts transitively on the set of ordered pairs (a,b), where $a,b\in\{\pm i,\pm j,\pm k\}$ and $a\neq b^{\pm1}$, we may assume that $x_2=i$, $x_3=j$. Therefore, $x_4=(x_1x_2x_3)^{-1}=k$.

Thus, every generating vector is equivalent to $(-1,i,j,k)$.

3. $|G|=10$.

3.1. $(5,10^2)$. Then $G\cong\mathbb{Z}_{10}$.

3.2. $(2^2,5^2)$. In this case, it is possible that $G\cong\mathbb{Z}_{10}$ and $G\cong D_5$.

4. $|G|=12$.

An arbitrary group of order 12 is either abelian or isomorphic to one of the groups D_6, $D_{4,3,-1}$, and A_4.

4.1. $(3,12^2)$, $(4,6,12)$. Then $G\cong\mathbb{Z}_{12}$.

4.2. (6^3). Then G is abelian, since each of the groups D_6 and $D_{4,3,-1}$ has a single subgroup isomorphic to $\mathbb{Z}_6$ (hence, in the cases $G\cong D_6$ and $G\cong D_{4,3,-1}$ the generation condition does not hold) and A_4 does not contain elements of order 6.

4.3. $(2^2,3,6)$. In this case, G is either abelian or isomorphic to D_6, since A_4 does not contain elements of order 6 and condition (3) does not hold in $D_{4,3,-1}$ in view of uniqueness of an involution.

4.4. (2^5). Then $G\cong D_6$, since subgroups generated by all involutions in an abelian group of order 12, in A_4, and in $D_{4,3,-1}$ are proper.

4.5. $(2,3^3)$. Since the Sylow 3-subgroups are uniquely determined in an abelian group of order 12, in D_6, and in $D_{4,3,-1}$ and, hence, condition (3) is violated, we have

$$G\cong A_4=\langle x,y \mid x=(12)(34),\ y=(123)\rangle.$$

Let $a=(x_1,x_2,x_3,x_4)$ be a generating $(2,3^3)$-vector for A_4. Prove that, for some vector (x_1',x_2',x_3',x_4') equivalent to a, two of the three Sylow 3-subgroups $\langle x_2'\rangle$, $\langle x_3'\rangle$, and $\langle x_4'\rangle$ coincide. Assume the contrary. Then, in particular, the subgroups $\langle x_2\rangle$, $\langle x_3\rangle$, and $\langle x_4\rangle$ are distinct.

Let P be the forth Sylow 3-subgroup in A_4. Consider the vectors $\theta_3a=(x_1,x_2,x_4,x_4^{-1}x_3x_4)$ and $\theta_1^{-2}a=(*,x_1x_2x_1^{-1},x_3,x_4)$. By assumption, we have $\langle x_4^{-1}x_3x_4\rangle\neq\langle x_2\rangle,\langle x_4\rangle$ and, moreover, $\langle x_4^{-1}x_3x_4\rangle\neq\langle x_3\rangle$, since $N_{A_4}(\langle x_3\rangle)=\langle x_3\rangle$.

Thus, $\langle x_4^{-1}x_3x_4\rangle=P$. Similarly, $\langle x_1x_2x_1^{-1}\rangle=P$. Then the second and the forth entries of the vector $\theta_3\theta_1^{-2}a=(*,x_1x_2x_1^{-1},x_4,x_4^{-1}x_3x_4)$ generate P, a contradiction.

Thus, without loss of generality, we may assume that two of the three entries x_2, x_3, and x_4 of the initial vector a generate the same Sylow 3-subgroup. Applying transformations of the form θ_j, we may assume that $\langle x_2\rangle=\langle x_3\rangle$. Since $x_1x_2x_3x_4=1$, $|x_1|=2$, and $|x_2|=|x_3|=|x_4|=3$, we have $x_2=x_3$. Next, the group $\mathrm{Aut}(A_4)$ acts transitively on the set of ordered pairs (g,h), where $|g| = 2$ and $|h| = 3$, $g, h \in A_4$. Therefore, the initial vector is equivalent to $(x, y, y, y^{-2}x)$.

4.6. $(2^2, 4^2)$. This case is impossible, since A_4 and D_6 never contain elements of order 4. The group $D_{4,3,-1}$ is rejected, since it has a unique involution and, hence, the generation condition and (3) do not hold simultaneously. An abelian group is not appropriate either.

5. $|G| = 15$.

Then $G \cong \mathbb{Z}_{15}$ and among the possible branching data $(3, 5, 15)$ and (5^3) only the first data are realized.

6. $|G| = 16$.

6.1. $(2, 16^2)$. Then $G \cong \mathbb{Z}_{16}$.

6.2. $(2^3, 8)$. Let (x_1, x_2, x_3, x_4) be a generating $(2^3, 8)$-vector for G. Then $\left|G : \langle x_4 \rangle\right| = 2$. Since $\langle x_1, x_2, x_3, x_4 \rangle = G$, we have $x_i \notin \langle x_4 \rangle$ for some $i \in \{1, 2, 3\}$. Using transformations of the form θ_j, we may assume that $x_1 \notin \langle x_4 \rangle$ and thus $G = \langle x_4 \rangle \leftthreetimes \langle x_1 \rangle$. The group G is nonabelian, since the product of elements of order 2 in an abelian group is not an element of order 8. The following three possibilities remain: $G \cong D_8$ and $G \cong D_{2,8,\pm 3} = \left\langle x, y \mid x^2 = y^8 = 1,\ x^{-1}yx = y^{\pm 3} \right\rangle$. The first can be considered with the help of Lemmas 3 and 4.

Consider the second and third possibilities. The group $\mathrm{Aut}(D_{2,8,\pm 3})$ acts transitively on the set of all ordered pairs (a, b) such that $\langle a, b \rangle = D_{2,8,\pm 3}$, $|a| = 2$, and $|b| = 8$. Therefore, we may assume that $x_1 = x$ and $x_4 = y$. Since the equality $x_1x_2x_3x_4 = 1$ is valid in all quotient groups of G, in particular in $G/\langle x_4 \rangle$, one of the elements x_2 or x_3 lies in $\langle x_4 \rangle$ and the remaining element lies outside of $\langle x_4 \rangle$. Applying transformations of the form $\theta_j^{\pm 1}$, we may assume that $x_2 \notin \langle x_4 \rangle$ and $x_3 \in \langle x_4 \rangle$. The only element of order 2 in the group $\langle x_4 \rangle = \langle y \rangle$ is y^4. Therefore, $x_3 = y^4$ and, hence, $x_2 = x_1^{-1}x_4^{-1}x_3^{-1} = xy^3$. However, $\left|xy^3\right| \neq 2$ in the groups $D_{2,8,\pm 3}$ and, thus, a generating $(2^3, 8)$-vector does not exist in these groups.

6.3. $(4^2, 8)$. Let (x_1, x_2, x_3) be a generating $(4^2, 8)$-vector for G. Since $\left|G : \langle x_3 \rangle\right| = 2$, we have $x_1^2 = x_3^4$ and $x_2^2 = x_3^4$. This fact, together with the equality $x_2 = x_1^{-1}x_3^{-1}$, implies that $x_1^{-1}x_3x_1 = x_3^{-1}$ and $G \cong \left\langle x, y \mid y^8 = 1,\ x^2 = y^4,\ x^{-1}yx = y^{-1} \right\rangle$. The group $\mathrm{Aut}(G)$ acts transitively on the set of ordered pairs (u, v) such that $\langle u, v \rangle = G$, $|u| = 4$, and $|v| = 8$. Therefore, we may assume that $x_1 = x$ and $x_3 = y$ and, hence, $x_2 = x^{-1}y^{-1}$.

7. $|G| = 18$.

7.1. $(2, 9, 18)$. Then $G \cong \mathbb{Z}_{18}$.

7.2. $(2^3, 3^2)$. Let (x_1, x_2, x_3, x_4) be a generating $(2^2, 3^2)$-vector for G. Let H be a Sylow 3-subgroup in G. By Proposition 3, H has the branching data (3^4) and, hence, $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Since $|x_1| = 2$, we obtain $G = H \leftthreetimes \langle x_1 \rangle$.

If $\langle x_1\rangle$ acts trivially on H then $G = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ and this case can be easily eliminated. If x_1 acts on H by inverting the elements then we consider this case with the help of Lemmas 3 and 4. In view of Lemma 1, it remains to examine the only case

$$G = \langle x, y \mid x^3 = y^3 = [x,y] = 1\rangle \leftthreetimes \langle B\rangle. \tag{11}$$

Let (x_1, x_2, x_3, x_4) be a generating $(2^2, 3^2)$-vector for G. Since the elements B, Bx, and Bx^2 are of order 2 in G and they are conjugate (by the elements $x^{\pm 1}$), we may assume that $x_1 = B$. Then $2 = |x_2| = |x_1^{-1}x_4^{-1}x_3^{-1}| = |Bx_4^{-1}x_3^{-1}|$ and, as a consequence, $x_3x_4 = x^i$ for some $i = 0,1,2$. The elements x_3 and x_4 are not in $\langle x\rangle$ simultaneously; otherwise, $\langle x_1, x_2, x_3, x_4\rangle = \langle x_1, x_3, x_4\rangle = \langle B, x\rangle < G$. By analogy, x_3 and x_4 are not in $\langle y\rangle$ simultaneously. The third case (one of the elements x_3 or x_4 lies in $\langle x\rangle$ and the other in $\langle y\rangle$) can be excluded too, since the condition $x_3x_4 = x^i$ is violated. Therefore, one of the elements x_3 or x_4 is equal to x^py^q, where $p, q \in \{-1, 1\}$. Applying the transformation θ_3^{-1} and the automorphism $\overline{\varphi}\colon x \to x^p$, $y \to y^q$, $B \to B$, we may assume that $x_4 = xy$ and, thus, $x_3 = x^{i-1}y^{-1}$. The transformation θ_2^{-2} substitutes $-(i-1)$ for $(i-1)$. Therefore, we have $x_3 = y^{-1}$ or $x_3 = xy^{-1}$, up to equivalence.

Thus, in this case, we obtain the following two vectors:

$$(B, Bx^{-1}, y^{-1}, xy) \quad \text{and} \quad (B, Bx, xy^{-1}, xy).$$

They are not equivalent, since the former vector has an entry lying in the center of G and the latter has not. Transformations of the form θ_j and automorphisms of the group G preserve the number of central elements in vectors.

7.3. $(2^3, 6)$. Prove that this case is impossible. Assume the contrary. Let (x_1, x_2, x_3, x_4) be a generating $(2^3, 6)$-vector for G. Then the normalizer of the involution x_4^3 either coincides with G or has index 3 in G. Therefore, the number of involutions in G is equal to 1 or 3. If there are identical involutions among x_1, x_2, and x_3, then obviously $|x_1x_2x_3| = 2$. The last equality contradicts the conditions $x_1x_2x_3x_4 = 1$ and $|x_4| = 6$. If x_1, x_2, and x_3 are three distinct involutions in G then the involutions x_1, $x_2^{x_1}$, and $x_3^{x_1}$ are also distinct. In this case, $x_2^{x_1} = x_3$ and $x_3^{x_1} = x_2$ (if $x_i^{x_j} = x_i$ for $i \neq j$ then $|\langle x_i, x_j\rangle| = 4$). Whence it follows readily that $|x_1x_2x_3| = 2$; a contradiction.

7.4. $(3, 6^2)$. As in 7.2, we can establish that $G = H \leftthreetimes \langle z\rangle$, where $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ is a Sylow 3-subgroup in G and z is an arbitrary involution in G. Then G is either abelian or of dihedral type or has the form (11). The first case can be considered directly. The second case is rejected, in view of the lack of elements of order 6 in such a group. Consider the third case.

Let (x_1, x_2, x_3) be a generating $(3, 6^2)$-vector for the group G in (11). All elements of order 6 in G have the form x^iy^jB, where $i = 0, 1, -1$ and

$j = -1, 1$. With the help of the automorphisms $\varphi_{ij}\colon B \to x^{-i}B,\ y \to y^j,\ x \to x$, they can be reduced to the element yB. Using φ_{ij} and transformations of the form θ_k, we may assume that $x_2 = yB$. Then the element x_3 of order 6 has the form $x^p y^q B$, where $p = \pm 1$ and $q = \pm 1$ (if $p = 0$ then $\langle x_1, x_2, x_3\rangle = \langle x_2, x_3\rangle = \langle y, B\rangle < G$, which is impossible).

Applying the automorphism $x \to x^{-1}$, $y \to y$, $B \to B$, we may derive that x_3 is equal to $xy^{\pm 1}B$. In this case, two generating $(3, 6^2)$-vectors, (xy, yB, xyB) and $(x, yB, xy^{-1}B)$, arise. They are not equivalent, since the first entry of the first vector is not an automorphic image of one of the entries of the second vector.

8. $|G| = 20$.

8.1. $(2^3, 5)$ and $(2, 10^2)$. Let $H \cong \mathbb{Z}_5$ be the only Sylow 5-subgroup in G and let L be some Sylow 2-subgroup in G. Then $G = H \rtimes L$. Using the method of Subsection 2.2 and our classification for the groups of order 4, we conclude that L has the branching data (2^7) or $(1 : 2^3)$ and, hence, $L \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Since at least one of the three involutions in L acts as identity on $H \cong \mathbb{Z}_5$, we have $G \cong \mathbb{Z}_{10} \rtimes \mathbb{Z}_2$. Then either $G \cong \mathbb{Z}_{10} \times \mathbb{Z}_2$ or $G \cong D_{10}$. The branching data for G are $(2, 10^2)$ in the former case and $(2^3, 5)$ in the latter.

8.2. $(4^2, 5)$. Then $G \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$ and either $G \cong \mathbb{Z}_{20}$ (which is immediately discarded in view of (3)) or $G \cong D_{4,5,-1} = \langle x, y \mid x^4 = y^5 = 1,\ x^{-1}yx = y^{-1}\rangle$ or $G \cong D_{4,5,2}$. Consider the second and third cases simultaneously.

Let (x_1, x_2, x_3) be an arbitrary generating $(4^2, 5)$-vector in G. With the help of $\mathrm{Aut}(G)$, we may assume that $x_3 = y$. Let $\bar{x}_1$ and $\bar{x}_2$ be the images of x_1 and x_2 in $G/\langle y\rangle$. It is easily seen that $|\bar{x}_1| = |\bar{x}_2| = 4$ and $\bar{x}_1\bar{x}_2 = 1$. Hence, $x_1 = y^i x^\varepsilon$ and $x_2 = y^j x^{-\varepsilon}$ for some i, j, and $\varepsilon \in \{-1, 1\}$. The transformation θ_1 replaces (in particular) ε with $-\varepsilon$. Therefore, we may assume that $x_1 = y^i x$. Applying the inversion of the automorphism $\varphi_i\colon x \to y^i x,\ y \to y$ to the vector (x_1, x_2, x_3), we obtain the $(4^2, 5)$-vector $(x, x^{-1}y^{-1}, y)$.

9. $|G| = 24$.

9.1. $(2^3, 4)$. Let G be a group of order 24 with a generating $(2^3, 4)$-vector (x_1, x_2, x_3, x_4). Since a group of order 24 is solvable, G has a normal subgroup of index 2 or 3. In the latter case, all the elements x_1, x_2, x_3, and x_4 are in the only Sylow 2-subgroup and do not generate G; this is impossible.

Let H be a subgroup of index 2 in G. In accord with Subsection 2.2 and our classification for groups of order 12, we conclude that either the branching data for H are (2^5) and $H \cong D_6$ or the branching data for H are $(1 : 2)$ and $H \cong A_4$.

In the former case, $H \cong D_6$ contains an automorphically invariant subgroup $L \cong \mathbb{Z}_3$. Therefore, $L \trianglelefteq G$, $|C_G(L)| = 12$, $C_G(L) \not\cong D_6$, and $C_G(L) \not\cong A_4$, a contradiction.

Consider the latter case, $H \cong A_4$. Since the involutions x_1, x_2, and x_3 generate G, one of them lies outside of H and, hence, $G \cong A_4 \rtimes \mathbb{Z}_2$. The case $G \cong A_4 \times \mathbb{Z}_2$ is impossible, since such a group has no elements of order 4. For the same reason, G is not isomorphic to a semidirect product of A_4 by means of an inner automorphism.

Thus, $G \cong S_4$. Since $|x_1x_2x_3| = 4$ and, hence, $x_1x_2x_3$ is an odd permutation, either the involutions x_1, x_2, and x_3 are transpositions or one of them is a transposition and the others are involutions in the subgroup $K = \{e, (12)(34), (13)(24), (14)(23)\}$. In the latter case, $\langle x_1, x_2, x_3\rangle$ is contained in a Sylow subgroup of S_4, which is impossible.

Next, all three transpositions x_1, x_2, and x_3 are distinct (otherwise, their product is a transposition) and contain the symbols 1, 2, 3, and 4 (otherwise, they do not generate S_4). With the use of transformations of the form θ_j, we can achieve that they have a common symbol. Indeed, if this is not true for the transpositions x_1, x_2, and x_3, then two of them have no common symbols. With the help of transformations of the form θ_j, we may assume that these elements are x_1 and x_2. Then x_3 has a common symbol with x_1 as well as with x_2. Applying θ_2, we obtain the involutions x_1, x_3, and $x_3^{-1}x_2x_3$ which have one common symbol. Finally, conjugating by an appropriate element of S_4, we may assume that $x_1 = (12)$, $x_2 = (13)$, and $x_3 = (14)$ and thus $x_4 = (4321)$.

9.2. $(3, 4, 6)$. This case was examined in the example of Subsection 2.2.

9.3. $(2, 6, 12)$. Let (x_1, x_2, x_3) be a generating $(2, 6, 12)$-vector for G. Since $|G : \langle x_3\rangle| = 2$, we have $x_2^2 \in \langle x_3\rangle$ and, therefore, $x_2^2 = x_3^{4\varepsilon}$, where $\varepsilon = \pm 1$. But $x_2 = x_1^{-1}x_3^{-1}$. Hence, $x_1^{-1}x_3^{-1}x_1^{-1}x_3^{-1} = x_3^{4\varepsilon}$ and $x_3^{x_1} = x_3^{-(4\varepsilon+1)}$. Since $|x_3| = 12$, the only case $\varepsilon = 1$ can occur. Thus,

$$G = \langle x_1, x_3 \mid x_1^2 = 1,\ x_3^{12} = 1,\ x_1^{-1}x_3x_1 = x_3^{-5}\rangle \cong D_{2,12,-5}.$$

Let now the group G be given abstractly as follows: $G = \langle x, y \mid x^2 = 1,\ y^{12} = 1,\ x^{-1}yx = y^{-5}\rangle$. Then every generating $(2, 6, 12)$-vector for G is automorphically equivalent to the vector (x, xy^{-1}, y).

9.4. The branching data $(2, 8^2)$, (4^3), and $(3^2, 12)$ are impossible. This can be proven with the help of the results of Subsection 2.2 and already known branching data for the groups of order 8 and for $\mathbb{Z}_{12}$.

10. $|G| = 32$, $(2, 4, 16)$.

As in case 9.3, we establish that $G \cong \langle x, y \mid x^2 = 1,\ y^{16} = 1,\ x^{-1}yx = y^7\rangle \cong D_{2,16,7}$ and every generating $(2, 4, 16)$-vector for G is equivalent to (x, xy^{-1}, y).

11. $|G| = 36$.

11.1. $(2^3, 3)$. Let (x_1, x_2, x_3, x_4) be an arbitrary generating $(2^3, 3)$-vector for G. Assume $H \leq G$ and L to be a Sylow 3-subgroup and a Sylow 2-subgroup, respectively. By Proposition 3 and the above classification for actions of groups of order 9 and 4, H has the branching data (3^4) and $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, L has the branching data (2^7) or $(1:2^3)$ and $L \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, Proposition 4 implies $H \trianglelefteq G$. The group L acts without kernel on H; otherwise, there exists a subgroup M in G isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$. In accord with our classification, such a group M has the branching data $(3, 6^2)$ and this fact contradicts Proposition 3.

Up to conjugation, $GL_2(3)$ has a single subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and this implies that

$$G = \langle x, y \mid x^3 = y^3 = [x, y] = 1 \rangle \leftthreetimes \langle B, A \mid B^2 = A^2 = [B, A] = 1 \rangle,$$

where $B^{-1}xB = x^{-1}$, $B^{-1}yB = y$, $A^{-1}xA = x^{-1}$, and $A^{-1}yA = y^{-1}$.

We need the following automorphisms of G:

$$\theta_x : x \to x^{-1}; \quad \theta_y : y \to y^{-1}; \quad \mu : A \to Ay; \quad \tau : B \to Bx, \quad A \to Ax.$$

Relative to the action of the group $\langle \theta_x, \theta_y, \mu, \tau \rangle$, the set of all involutions in G is decomposed into the following three orbits: $\{Ax^i y^j\}$, $\{BAy^q\}$, and $\{Bx^p\}$, where i, j, p, and q range over the set $\{0, 1, -1\}$. The product of two involutions in the same orbit lies in $\langle x, y \rangle$.

We claim that the involutions x_1, x_2, and x_3 belong to distinct orbits. Indeed, if, for example, x_1 and x_3 belong to the same orbit then $x_2 = x_1 x_3 \left(x_4^{-1}\right)^{x_3} \in \langle x, y \rangle$, which is impossible. Applying transformations of the form θ_j we may assume that x_1, x_2, and x_3 are involutions of the first, second, and third orbit, respectively. Next, the automorphism $\tau^{-i}\mu^{-j}$ sends the vector (x_1, x_2, x_3, x_4) to an equivalent generating vector of the form $(A, BAy^n, Bx^m, x^{-m}y^{-n})$. Observe that $n \neq 0$ and $m \neq 0$; otherwise, this vector is not generating. With the help of the automorphisms θ_x and θ_y, from this vector we can obtain the vector $(A, BAy, Bx, x^{-1}y^{-1})$.

11.2. $(2, 6^2)$. Let (x_1, x_2, x_3) be a generating $(2, 6^2)$-vector for G. As in case 11.1, we establish that $G = H \leftthreetimes L$, where $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $L \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We consider three subcases.

(A) Let the subgroup L act without kernel on H. Then the group G is the same as that in 11.1. There are two types of elements of order 6 in G: $Bx^i y^j$ and $BAx^j y^i$, where $i = 0, -1, 1$ and $j = -1, 1$. Since the elements of the same type do not generate G, we conclude that x_2 and x_3 are of different types. Applying θ_2 if necessary, we may assume that x_2 and x_3 are elements of the first and second type, respectively. With the use of the automorphisms θ_x, θ_y, μ, and τ, we can transform any such pair (x_2, x_3) into the pair (By, BAx) and obtain the generating $(2, 6^2)$-vector (Axy^{-1}, By, BAx).

(B) Assume existence of two involutions t and z in L such that the first one acts identically on H and the second does not. Then $G = \left(H \leftthreetimes \langle z \rangle\right) \times \langle t \rangle$, where $H = \langle x, y \mid x^3 = y^3 = [x, y] = 1 \rangle$, and, in view of Lemma 1, either

$z^{-1}xz = x^{-1}$ and $z^{-1}yz = y^{-1}$ or $z^{-1}xz = x^{-1}$ and $z^{-1}yz = y$. In the first case, all elements of order 6 (in particular, x_2 and x_3) lie in $H \times \langle t \rangle$ and, thus, do not generate G. Therefore, only the second case can be realized and, in this case, $G \cong S_3 \times \langle u \mid u^6 = 1 \rangle$.

Note that x_1 is a noncentral involution in G; otherwise, $\langle x_1, x_2 \rangle \neq G$. Since all noncentral involutions in G are automorphically conjugated, we may assume that $x_1 = (12)$. Next, the equalities $\langle x_1, x_2 \rangle = G$ and $\langle x_1, x_3 \rangle = G$ imply that x_i (for $i = 2, 3$) are of the form $a_i u^{\varepsilon_i}$, where $a_i \neq e$ and $\varepsilon_i \in \{-1, 1\}$.

Since $x_1 x_2 x_3 = 1$, either a_2 or a_3 is a triple cycle. Employing the transformation θ_2, we may assume that a_2 is a triple cycle and, using $\mathrm{Aut}(G)$, we may assume that $x_2 = (123)u$. In this case, $\big((12), (123)u, (13)u^{-1}\big)$ is a generating $(2, 6^2)$-vector for G.

(C) Let L act as identity on H. Then $G \cong \mathbb{Z}_6 \times \mathbb{Z}_6$, which is clearly impossible.

11.3. $(3, 4^2)$. Let (x_1, x_2, x_3) be a generating $(3, 4^2)$-vector for G, let H be a Sylow 3-subgroup in G, and let $L = \langle x_3 \rangle \cong \mathbb{Z}_4$ be a Sylow 2-subgroup. As in 11.1, we can establish that $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $H \trianglelefteq G$. Prove that L acts without kernel on H. Assume the contrary. Then the element x_3^2 lies in the kernel of the action and, hence, in the center of G. The number of its fixed points is equal to

$$\left|T_4^{x_3^2}\right| = \left|N_G(\langle x_3^2 \rangle)\right| \left(\frac{\delta_1}{3} + \frac{\delta_2}{4} + \frac{\delta_3}{4}\right) = 36\left(\frac{1}{4} + \frac{1}{4}\right) = 18$$

(see the definition of δ_i in Subsection 2.3 with regard to the fact that $\langle x_2 \rangle$ is conjugate to $\langle x_3 \rangle$). However, an element of order 2 can have at most ten fixed points (see Remark (8) and the classification for actions of groups of order 2), a contradiction.

Thus, L acts without kernel on H. Since $GL_2(3)$ contains one conjugacy class of elements of order 4, we have

$$G = \big\langle x, y, C \;\big|\; x^3 = y^3 = [x, y] = 1,\\ C^4 = 1,\ C^{-1}xC = y^{-1},\ C^{-1}yC = x \big\rangle.$$

Moreover, we may assume that $x_3 = C$. All elements of order 4 in G have the form Cx^iy^j or $C^{-1}x^iy^j$, where $i, j \in \{0, -1, 1\}$. The element x_2 is of the form $C^{-1}x^iy^j$; otherwise, the element $x_1 = (x_2x_3)^{-1} = x^{-j}y^iC^2$ is of order 2. In this case, $x_2 \neq C^{-1}$; otherwise, x_1 is equal to 1. Consider the automorphism $\alpha\colon x \to xy,\ y \to x^{-1}y,\ C \to C$. It is of order 8 and acts transitively on the set $\langle x, y \rangle \setminus \{1\}$. Applying an appropriate power of α to $(x_1, C^{-1}x^iy^j, C)$, we obtain the vector $\big(x_1', C^{-1}x, C\big)$, where $x_1' = y$ by (3).

11.4. $(3^2, 6)$. Since the group G of order 36 is solvable, it contains a normal subgroup H of index 2 or 3. We can discard the first case in view of

Subsection 2.2 and our classification. Therefore, we assume that $H \trianglelefteq G$ and $|G:H| = 3$. If (with the notation of Proposition 3) $\tau = 0$ then, by Propositions 3 and 4 together with our classification, either H has the branching data (6^3) and, hence, $H \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ or H has the branching data $(2, 3^3)$ and, hence, $H \cong A_4$. If $\tau > 0$ then H has the branching data $(1:2)$ and $H \cong A_4$.

In any case, a Sylow 2-subgroup in H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and automorphically invariant. It is readily seen that a Sylow 3-subgroup in G is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Hence, $G \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \leftthreetimes (\mathbb{Z}_3 \times \mathbb{Z}_3)$ and the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ acts on $\mathbb{Z}_2 \times \mathbb{Z}_2$ with a nontrivial kernel M. Observe that $M \ncong \mathbb{Z}_3 \times \mathbb{Z}_3$, since the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ cannot act on T_4 with branching data $(3^2, 6)$ (it is not generated by the elements of order 3). Therefore, $M \cong \mathbb{Z}_3$ and $G = A_4 \times \langle x \mid x^3 = 1 \rangle$.

Since all elements of order 6 in G are automorphically conjugate to $(12)(34)x$, we may assume that $x_3 = (12)(34)x$. In view of the equalities $|x_1| = 3$ and $\langle x_1, x_3 \rangle = G$, we have $x_1 = (ijk)x^l$, $l \in \{0, 1, -1\}$. Applying the automorphism $\overline{\varphi} \in \mathrm{Aut}(G)$ such that $\overline{\varphi}(x) = x$, $\overline{\varphi}\big((12)(34)\big) = (12)(34)$, and $\overline{\varphi}\big((ijk)x^l\big) = (123)$, we may assume that $x_1 = (123)$. Thus, a generating $(3^2, 6)$-vector for G is equivalent to $\big((123), (143)x^{-1}, (12)(34)x\big)$.

11.5. $(2, 4, 12)$. No group with such branching data exists; it suffices to apply the results of Subsection 2.2 to a subgroup generated by an element of order 12, taking account of our classification.

12. $|G| = 40$, $(2, 4, 10)$. Let (x_1, x_2, x_3) be a generating $(2, 4, 10)$-vector for G, let H be the unique Sylow 5-subgroup in G, and let L be some Sylow 2-subgroup in G. Then $G = H \leftthreetimes L$. We state that $L \cong D_4$.

Indeed, if $L \ncong D_4$ then our classification for actions of groups of order 8 yields $L \cong \mathbb{Z}_8$ or $L \cong Q_8$. Since $\mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$, the subgroup L acts on H with a nontrivial kernel. This kernel contains the only involution in L. Then this involution belongs to the center of G and it is the only involution in G too. In much the same way as in 11.3, formula (7) yields a contradiction.

Suppose that

$$H = \langle x \mid x^5 = 1 \rangle, \quad L = \langle y, z \mid y^4 = 1,\ z^2 = 1,\ z^{-1}yz = y^{-1} \rangle.$$

The element y acts nontrivially on H (otherwise, $|xy| = 20$, which contradicts Proposition 6). Substituting zy or zy^2 for z if necessary, we may assume that z centralizes H. In this case, the relation $z^{-1}yz = y^{-1}$ implies that y acts on H as an automorphism of order 2. Therefore,

$$G = \langle x, y, z \mid x^5 = 1,\ y^4 = 1,\ z^2 = 1,\ z^{-1}yz = y^{-1},\ y^{-1}xy = x^{-1},\ z^{-1}xz = x \rangle.$$

We need the following automorphisms of G:

$$\mu_i \colon x \to x^i\ (i = 1, 2, 3, 4) \quad \text{and} \quad \tau \colon z \to zy^2.$$

By the Sylow theorem, an element of order 4 in G is conjugate to y. Therefore, assume $x_2 = y$. Since $\langle x_1, x_2\rangle = G$, we have $x_1 \notin L$. All involutions of G that do not lie in L have the form $x^i z y^\varepsilon$ ($i = 1,2,3,4$; $\varepsilon = -1,1$). The automorphism μ_i^{-1} or $\mu_i^{-1}\tau$ transforms them into xzy (note that $x_2 = y$ is a fixed point). Therefore, we may assume that $x_1 = xzy$ and $x_3 = (x_1x_2)^{-1} = y^2zx^{-1}$. Finally, we arrive at the generating $(2,4,10)$-vector (xzy, y, y^2zx^{-1}).

13. $|G| = 60$.

13.1. $(2,5^2)$. Assume that G is not a simple group. Then there exists a normal subgroup in G of index 2 or 3 or 5. The first case is impossible due to Lemma 8. The second case is not suitable too in view of Subsection 2.2 and the above-listed branching data for groups of order 20.

Assume now existence of an $H \trianglelefteq G$, $|H| = 12$. Taking into account the branching data for groups of order 12 and Subsection 2.2, we conclude that H has the branching data $(1 : 2)$ or (2^5). With regard to our classification, $H \cong A_4$ in the first case and $H \cong D_6$ in the second case. Let L denote the Klein subgroup in H in the former case and the cyclic subgroup of order 6 in H in the latter case. In both cases, L is a characteristic subgroup in H, $L \trianglelefteq G$. Let M be a Sylow 5-subgroup in G. Then $L \leftthreetimes M$ is a normal subgroup in G of index 3 in the first case and of index 2 in the second case. Both situations were eliminated.

Thus, G is a simple group and, hence, $G \cong A_5$. Let (x_1, x_2, x_3) be a generating $(2,5^2)$-vector for G. Applying an appropriate automorphism, we may assume that $x_2 = (12345)$. Observe that the Sylow 5-subgroups generated by x_2 and x_3 are distinct because of the equality $\langle x_2, x_3\rangle = G$. Since x_2 acts by conjugation on the six Sylow 5-subgroups, stabilizing $\langle x_2\rangle$ and interchanging the remaining five cyclically, we may assume that $x_3 \in \langle(13452)\rangle$. The element $x_3 = (13452)$ is the only element such that $x_1^{-1} = x_2x_3$ is of order 2. Thus, we obtain the generating $(2,5^2)$-vector $\big((24)(35), (12345), (13452)\big)$.

13.2. $(2,3,15)$. Prove that groups of order 60 with these branching data do not exist. Assume the contrary. Then G is not a simple group, because A_5 has no elements of order 15. Therefore, G has a normal subgroup of one of the indices 2, 5, or 3. The first case can be excluded by Lemma 8, and the second case can be excluded in view of Subsection 2.2 and the above-listed branching data for the groups of order 12.

Assume now that $H \trianglelefteq G$ and $|G : H| = 3$. Taking account of Subsection 2.2 and the above-listed branching data for the groups of order 20, we obtain the branching data $(2^3, 5)$ for H. In accord with our classification, we have $H \cong D_{10}$.

Assume L to be a characteristic subgroup in H of order 10 and M to be a Sylow 3-subgroup in G. Then $L \trianglelefteq G$ and $L \leftthreetimes M$ is a subgroup of index 2 in G, which has already been eliminated.

14. $|G| = 72$.

In this case, G has a normal subgroup of index 2 or 3.

14.1. $(3^2, 4)$. There are no such groups in view of Subsection 2.2 and our classification of branching data for groups of order 24 and 36.

14.2. $(2, 3, 12)$. Suppose that G contains a normal subgroup M of index 3. In view of our classification and Subsection 2.2, M has the branching data $(2^3, 4)$ and, hence, $M \cong S_4$. Since a perfect normal subgroup is always a direct factor of a group, we have $G = M \times C_G(M)$. Therefore, both G and M contain a subgroup of index 2.

Thus, we need to consider the case in which G contains a subgroup H of index 2. In view of our classification and Subsection 2.2, H has the branching data $(3^2, 6)$ and $H \cong A_4 \times \mathbb{Z}_3$. Since the branching data for G are $(2, 3, 12)$, G is generated by some elements of order 2 or 3. Hence, $G \cong H \leftthreetimes \mathbb{Z}_2$.

Let $G = \big(A_4 \times \langle x\rangle\big) \leftthreetimes \langle y\rangle$, where $\langle x\rangle \cong \mathbb{Z}_3$, $\langle y\rangle \cong \mathbb{Z}_2$, and $p\colon A_4 \times \langle x\rangle \to A_4$ is the projection onto the first factor. Define a homomorphism $\tau\colon A_4 \to A_4$ by the equality $\tau(a) = p(y^{-1}ay)$ for $a \in A_4$. This homomorphism is an automorphism, since the Klein subgroup K is y-invariant. Next, $\langle x\rangle \trianglelefteq G$, since $\langle x\rangle$ is the center of H. This implies that $|\tau| = 2$.

By our classification, $K \leftthreetimes \langle y\rangle \cong D_4$ and, hence, τ (as well as y) stabilizes one of the involutions in K and interchanges the other two. Therefore, the action of the involution τ on A_4 coincides with the conjugation of A_4 by some transposition in S_4. Renumbering the set $\{1, 2, 3, 4\}$, we may assume that this transposition is (12). Next, two subcases are possible.

(a) $A_4^y = A_4$. Then $\langle A_4, y\rangle \cong S_4$. By our classification, the group S_4 has the branching data $(2^3, 4)$ and $\langle A_4, y\rangle \trianglelefteq G$ in view of Subsection 2.2. Since $\langle x\rangle \trianglelefteq G$, we have $G = \langle A_4, y\rangle \times \langle x\rangle \cong S_4 \times \mathbb{Z}_3$.

(b) $A_4^y \neq A_4$. Then $(123)^y = (123)^{(12)}x^{\varepsilon}$, where $\varepsilon = \pm 1$. The equalities $|y| = 2$ and $|x| = 3$ imply $x^y = x$ and $\big((123)x^{\varepsilon}\big)^y = \big((123)x^{\varepsilon}\big)^{-1}$. Therefore, we obtain

$$G = \big(\big\langle (12)(34), (123)x^{\varepsilon}\big\rangle \leftthreetimes \langle y\rangle\big) \times \langle x\rangle \cong S_4 \times \mathbb{Z}_3.$$

Thus, in both subcases, we may assume that

$$G = S_4 \times \langle x \mid x^3 = 1\rangle.$$

Let (x_1, x_2, x_3) be a generating $(2, 3, 12)$-vector for G. In this case, $|x_1| = 2$ and $x_1 \in S_4$. Since $\langle x_1, x_2\rangle = G$ and $|x_2| = 3$, the elements x_1 and x_2 have the following form: $x_1 = (rs)$ and $x_2 = (ijk)x^{\pm 1}$. Employing $\mathrm{Aut}(G)$, we may assume that $x_1 = (14)$ and $x_2 = (123)x$. Thus, we arrive at the generating $(2, 3, 12)$-vector $\big((14), (123)x, (1324)x^{-1}\big)$.

14.3. $(2, 4, 6)$. Let H be a Sylow 3-subgroup in G and let L be a Sylow 2-subgroup in G. In view of Subsection 2.2 and our classification, H has the branching data (3^4), $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, and $H \trianglelefteq G$. Since L contains an element of order 4, the group L is isomorphic to one of the groups $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_8$, Q_8, or D_4. The group $\mathbb{Z}_4 \times \mathbb{Z}_2$ is absent in our classification. The possible branching data for the groups $\mathbb{Z}_8$ and Q_8 are $(2^2, 8^2)$ and $(2, 4^3)$, respectively; by Subsection 2.2, these groups are also excluded. Therefore, $L \cong D_4$.

Prove that L acts without kernel on H. If the kernel of the action is not trivial then it contains a central involution g of L. Then

$$|T_4^g| = |N_G(\langle g\rangle)|\left(\frac{\delta_1}{2} + \frac{\delta_2}{4} + \frac{\delta_3}{6}\right)$$

(see Subsection 2.3) and either $|T_4^g| = 0$ (if $\delta_i = 0$ for all i) or $|T_4^g| \geq 12$ (if there exists an i such that $\delta_i = 1$). However, an involution can only have 2 or 6 or 10 fixed points (see the classification and Remark (8)), a contradiction.

By Lemma 1, the group $\mathrm{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong GL_2(3)$ has only one conjugacy class of subgroups isomorphic to D_4. Therefore,

$$\begin{aligned} G = \Big\langle x, y, C, B \mid\ & x^3 = y^3 = [x, y] = 1, \\ & C^4 = B^2 = 1,\ B^{-1}CB = C^{-1},\ C^{-1}xC = y^{-1}, \\ & C^{-1}yC = x,\ B^{-1}xB = x^{-1},\ B^{-1}yB = y\Big\rangle. \end{aligned}$$

The group G has the following automorphisms: $\tau\colon x \to x^{-1}$, $C \to C^{-1}$; $\mu\colon C \to Cy$, $B \to Bx$; and $\nu\colon x \to xy^{-1}$, $y \to xy$, $B \to BC$.

Let (x_1, x_2, x_3) be a generating $(2,4,6)$-vector for G. Each element of order 4 in G is of the form $C^\varepsilon x^i y^j$, where $\varepsilon = \pm 1$ and $i, j \in \{0, 1, -1\}$. Applying to it the automorphisms $\mu^{i-j}\widehat{Cy^i}$ for $\varepsilon = 1$ and $\tau\mu^{-i-j}\widehat{Cy^{-i}}$ for $\varepsilon = -1$, we obtain C. Therefore, we may assume that $x_2 = C$.

An element of order 2 in G belongs to the union of the sets

$$\{Bx^i, BCx^iy^{-i}, BC^2y^i, BC^3x^iy^i \mid i \in \{0, 1, -1\}\}$$

and

$$\{C^2x^iy^j \mid i, j \in \{0, 1, -1\}\}.$$

To ensure that $\langle x_1, x_2\rangle = G$, we need to show that the element x_1 belongs to the first set and $i \neq 0$. All elements of the first set are automorphically conjugated for $i \neq 0$; this fact can be easily verified by using the automorphisms ν and $\widehat{C^2}$ that leave $x_2 = C$ invariant.

Thus, we may assume that $x_1 = Bx$ and thus obtain the generating $(2,4,6)$-vector (Bx, C, BCx).

15. $|G| = 120$, $(2, 4, 5)$.

Since a group of order 120 is not simple, the group G contains a proper normal subgroup H of minimal index and G/H is isomorphic to one of the following groups: $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, or A_5. Consider each of these cases separately.

(a) $G/H \cong \mathbb{Z}_2$. In this case, taking account of the classification of actions for groups of order 60, we infer $H \cong A_5$. In what follows, we assume that $H = A_5$.

Let (x_1, x_2, x_3) be a generating $(2,4,5)$-vector for G. Since $x_3 \in H$ and $\langle x_1, x_3\rangle = G$, we have $G = H \leftthreetimes \langle x_1\rangle$. The element x_1 acts without kernel on H; otherwise, $G = A_5 \times \mathbb{Z}_2$ and G has no elements of order 4 (on the other hand, $x_2 \in G$ and $|x_2| = 4$). Since $\mathrm{Aut}(A_5) = S_5$, x_1 acts on $H = A_5$ as conjugation by some involution a in S_5. If $a \in A_5$ then the involution ax_1 acts trivially on A_5 and $G = A_5 \times \langle ax_1\rangle$. But the last equality has already been discarded. Therefore, a is a transposition in S_5 and $G = H \leftthreetimes \langle x_1\rangle \cong S_5$.

Thus, we assume that $G = S_5$, x_1 is a transposition, and x_3 is a cycle of length 5. Conjugating by an appropriate element in S_5, we may assume that $x_3 = (12345)$ and $x_1 = (1j)$ for some $j \in \{2,3,4,5\}$. The condition that $x_2 = x_1^{-1}x_3^{-1}$ is an element of order 4 is satisfied only for $j = 2$ and $j = 5$. But $(12) = (15)^{x_3}$. Therefore, we may assume that $x_1 = (12)$ and thus arrive at the generating $(2,4,5)$-vector $\big((12), (2543), (12345)\big)$.

(b) $G/H \cong \mathbb{Z}_3$. In view of Subsection 2.2 and the known branching data for groups of order 40, this case is impossible.

(c) $G/H \cong \mathbb{Z}_5$. This case can be eliminated by analogy with case (b).

(d) $G/H \cong A_5$. Then either $G \cong A_5 \times \mathbb{Z}_2$ or $G \cong SL_2(5)$. The first subcase has already been eliminated. Let $G \cong SL_2(5)$. Since $SL_2(5)$ contains only one involution, we have $\langle x_1, x_2\rangle = \langle x_2\rangle \neq G$, a contradiction.

16. $|G| = 144$, $(2,3,8)$.

Prove that there are no such groups. Otherwise, G has a normal subgroup of index 2 or 3. The former case contradicts Subsection 2.2 and the above-described branching data for groups of order 72. In view of Lemma 8, the latter case is eliminated too.

We have settled all cases and so the proof of the theorem is complete.

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