Homotopical Fibring

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Over the last fifteen years the problem of fibring, i.e. of combining several logics into a single common extension, has received a lot of attention, both for its significance in applications and its interest for theoretical logic, see e.g. [3], [5], [6]. The main issues about fibring that have been studied in the literature are fibring techniques for different settings of logic — like abstract consequence relations, institutions or logics with frame semantics or topos semantics — and the preservation of metaproperties of logics under fibring — e.g. the existence of implicit connectives [3], completeness [2], or the position in the Leibniz hierarchy (which, roughly, measures the degree of algebraizability/applicability of the Lindenbaum-Tarski algebra technique) [7].

A major conceptual advance, in [1], see also [10], was the recognition of fibring as a colimit construction in an appropriate category of logics and translations between them. In particular the combination of two logics sharing a common sublogic, called their constrained fibring, was shown to be the pushout of the two logics along the inclusions of the sublogic into both of them.

However, the categories in which these considerations take place have a very restricted notion of morphism: Morphisms are translations mapping the primitive connectives, which generate the domain language, to primitive connectives of the target language. A more natural notion of morphism would be to allow primitive connectives to be mapped to derived connectives, as it happens e.g. in the $\neg\neg$-translations from intuitionistic to classical logic. Unfortunately the categories of logics with this broader notion of morphism are badly behaved, in particular colimits other than the ones from the old setting do either not exist at all or are degenerate and do not describe a combination of logics as desired. It has thus been an open problem how to combine logics along more general translations.

In this talk we present a solution: Any category of logics known to the speaker comes with a natural definition of when a translation is an equivalence of logics. It is thus open to the methods of abstract homotopy theory, e.g. those exposed in [12] or [13] — in particular the notion of homotopy colimit is defined, and this is what we call the *homotopical fibring*, or *hofibring*, of logics, and what we propose to replace the colimit construction of fibring with. The main conceptual advantages of hofibring over fibring in this setting are:

- homotopy colimits tend to exist in settings of interest where colimits do not exist
- one can always see the constituent logics as linguistic fragments of their hofibring, unlike for fibring
- invariance under equivalence: replacing the logics to be combined by equivalent ones will result in an equivalent hofibred logic

As an example we present the concrete meaning of this in a simple setting of propositional Hilbert Calculi. In this setting we consider signatures S of generating connectives given with arities (formally: a map $S \to \mathbb{N}$) and the absolutely free S-algebra $L(S)$ generated by a fixed set of propositional variables.

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We now work in the category whose objects are logics, presented by pairs $(S, \vdash)$, where S is a signature and $\vdash \subseteq \mathscr{P}(L(S)) \times L(S)$ is a consequence relation between subsets of $L(S)$ and elements of $L(S)$. As usual we write the consequence relation in infix notation $\Gamma \vdash \varphi$ and say that Γ entails φ. A morphism $f : (S, \vdash) \to (S', \vdash')$ is an arity preserving map $f : S \to L(S')$ such that its extension $\hat{f} : L(S) \to L(S')$ to the absolutely free algebra $L(S)$ satisfies $\Gamma \vdash \varphi \Rightarrow \hat{f}(\Gamma) \vdash' \hat{f}(\varphi)$ (i.e. it is a "translation").

There is an obvious forgetful functor from logics to signatures and it is easy to see that it preserves and creates colimits. Thus the underlying signature of a colimit of logics will always be the colimit of their underlying signatures. If, as in our case, signatures are given by free algebras, this explains the problem of defective or non-existing colimits: Colimits of free algebras are not "by nature" free again, and forcing them to be so results in degeneracy.

One can call a translation $f : (S, \vdash) \to (S', \vdash')$ a *weak equivalence*, if $\Gamma \vdash \varphi \Leftrightarrow \hat{f}(\Gamma) \vdash \hat{f}(\varphi)$ (i.e. it is a "conservative translation") and if for every formula φ in the target there exists a formula in the image of $\hat{f}$ which is logically equivalent to φ (it has "dense image").

One can further call *cofibration* a morphism which is given by mapping generating connectives injectively to generating connectives (i.e. which is given by an injective, arity preserving map between the signatures).

This category of logics and translations now has the convenient structure of a so–called ABC cofibration category (see [11]), that is, we have a factorization of any morphism into a cofibration followed by a weak equivalence, satisfying some axioms. The proof of this proceeds in close analogy to the original topological setting, e.g. by constructing mapping cylinders. The results of [11] then yield a concrete construction recipe for the homotopy colimit of a given diagram as the actual colimit of a different diagram, by which we can

- express hofibring through fibring
- see that fibring is a special case of hofibring (which yields a new universal property of fibring)
- see that all homotopy colimits exist and
- transfer preservation results known from fibring to hofibring, for metaproperties which are invariant under equivalence

Among the preservation results obtained by the technique of the last point, those on the existence of implicit connectives are straightforward. Preservation of completeness and position in the Leibniz hierarchy require a homotopical view on semantics as well, which we will provide before applying the transfer result.

In the talk we will make the above statements concrete in examples. To conclude, we will say how the above results extend to first order logics by admitting many-sorted signatures and to the fibring of institutions via the c-parchments from [14]. Finally we sketch a picture of homotopical versions of other variants of fibring, like modulated fibring ([4]), metafibring ([8]) and fibring of non-truth functional logics [9], as well as of the homotopical categories given by other settings like institutions and type theory. If time remains, we hint at a variety of approaches to abstract logic suggested by the homotopical view point.

References

[1] A. Sernadas, C. Sernadas, and C. Caleiro. Fibring of logics as a categorial construction. Journal of Logic and Computation, 9(2):149–179, 1999.

[2] A. Zanardo, A. Sernadas, and C. Sernadas. Fibring: Completeness preservation. Journal of Symbolic Logic, 66(1):414–439, 2001.

[3] W. A. Carnielli, M. E. Coniglio, D. Gabbay, P. Gouveia, and C. Sernadas. Analysis and Synthesis of Logics - How To Cut And Paste Reasoning Systems, volume 35 of Applied Logic. Springer, 2008

[4] C. Sernadas, J. Rasga, and W. A. Carnielli. Modulated fibring and the collapsing problem. Journal of Symbolic Logic, 67(4):1541–1569, 2002.

[5] Dov M. Gabbay. Fibring Logics. Clarendon Press, 1999

[6] W.A. Carnielli; M.E. Coniglio. Combining Logics. In: The Stanford Encyclopedia of Philosophy, 2007. Ed.: E.N. Zalta. URL: http://plato.stanford.edu/entries/logic-combining/

[7] V.L. Fernández; M.E. Coniglio. Fibring in the Leibniz Hierarchy. Logic Journal of the IGPL 15, no. 5-6: 475-501, 2007.

[8] M.E. Coniglio. Recovering a logic from its fragments by meta-fibring. Logica Universalis 1, no. 2: 377-416, 2007.

[9] C. Caleiro; W. Carnielli; M.E. Coniglio; A. Sernadas; C. Sernadas. Fibring Non-Truth-Functional Logics: Completeness Preservation. Journal of Logic, Language and Information 12, no. 2: 183-211, 2003.

[10] C. Caleiro, W. A. Carnielli, J. Rasga, and C. Sernadas. Fibring of logics as a universal construction. In D. Gabbay and F. Guenthner, editors, Handbook of Philosophical Logic, 2nd Edition, volume 13, pages 123–187. Springer, 2005.

[11] A. Radulescu-Banu, Cofibrations in Homotopy Theory, URL: http://arxiv.org/abs/math/0610009

[12] J. Lurie, Higher Topos Theory, Annals of Mathematics Studies 170, Princeton University Press, 2009

[13] C. Barwick, D. M. Kan, Relative categories: Another model for the homotopy theory of homotopy theories, URL: http://arxiv.org/abs/1011.1691

[14] C. Caleiro, P. Mateus, J. Ramos, and A. Sernadas. Combining logics: Parchments revisited. In M. Cerioli and G. Reggio, editors, Recent Trends in Algebraic Development Techniques - Selected Papers, volume 2267 of Lecture Notes in Computer Science, pages 48–70. Springer, 2001.