Lindström's Theorem

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The statement

Theorem (Lindström)

There is no logic that is more expressive than classical first order logic and that satisfies both the Compactness and the Löwenheim-Skolem properties.

From: Per Lindström, On extensions of elementary logic, Theoria 35, p.1-11, 1969

II. The proof

following Ebbinghaus/Flum/Thomas, Introduction to mathematical logic, Chapters XII/XIII

1st outline of Lindström's proof: Let $\mathcal{L}$ be a regular logic satisfying $\mathrm{LöSko}(\mathcal{L})$ and $\mathrm{Comp}(\mathcal{L})$. Assume that $\mathcal{L}_{\omega\omega} < \mathcal{L}$.

Then there exists a $\psi \in L(S)$ not equivalent to any first order sentence.

1. Show that for all $m \in \mathbb{N}$ there exist S-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \models_{\mathcal{L}} \psi$, $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$ and $\mathfrak{A} \cong_m \mathfrak{B}$.

2. Using $\mathrm{Comp}(\mathcal{L})$ we get p-isomorphic S-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \models_{\mathcal{L}} \psi$, $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$.

3. By $\mathrm{LöSko}(\mathcal{L})$ we can assume w.l.o.g. that $\mathfrak{A}$ and $\mathfrak{B}$ are *countable*. Then we have $\mathfrak{A} \cong \mathfrak{B}$ but $\mathfrak{A} \models_{\mathcal{L}} \psi$, $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$. This contradicts the isomorphism invariance of abstract logics!

Let $\mathcal{L}$ be a regular logic satisfying $\mathrm{LöSko}(\mathcal{L})$ and $\mathrm{Comp}(\mathcal{L})$. Assume that $\mathcal{L}_{\omega\omega} < \mathcal{L}$.

Then there exists a $\psi \in L(S)$ not equivalent to any first order sentence.

By $\mathrm{Repl}(\mathcal{L})$ (allowing to replace function symbols with relation symbols) we can assume w.l.o.g. that S contains only relation symbols. Remember that we want to prove the following:

Proposition(Step 1)**:** For all $m \in \mathbb{N}$ and all finite $S_0 \subseteq S$ there exist S-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \models_{\mathcal{L}} \psi$, $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$ and $\mathfrak{A}|_{S_0} \cong_m \mathfrak{B}|_{S_0}$.

(Note that we had to pass to a finite subsignature for our m-isomorphism. This will not be a problem in the course of the proof of Lindström's theorem)

Proposition(Step 1)**:** For all $m \in \mathbb{N}$ and all finite $S_0 \subseteq S$ there exist S-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \models_{\mathcal{L}} \psi$, $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$ and $\mathfrak{A}|_{S_0} \cong_m \mathfrak{B}_{S_0}$.

Proof: Let $S_0 \subseteq S$ be finite and $m \in \mathbb{N}$. Define $\varphi := \bigvee\{\varphi^m_{\mathfrak{A}|_{S_0},\emptyset} \mid \mathfrak{A} \models \psi\}$ ("this structure is m-isomorphic to $\mathfrak{A}|_{S_0}$ for an $\mathfrak{A}$ with $\mathfrak{A} \models \psi$")

There is an $\mathfrak{A}$ with $\mathfrak{A} \models \psi$, because otherwise ψ would be logically equivalent to a contradiction (this is what it means to have no models), and hence a first order formula. So the above disjunction is non-empty. From the definition of the $\varphi^m_{\mathfrak{A}|_{S_0},\emptyset}$ one can also see that the disjunction is finite, so φ is a first order sentence.

Clearly $\psi \rightarrow \varphi$ is a valid formula: If an S-structure $\mathfrak{A}$ satisfies $\mathfrak{A} \models \psi$, then it occurs in the disjunction and its S_0-reduction is m-isomorphic to itself.

On the other hand $\varphi \rightarrow \psi$ (i.e. $\neg\varphi \vee \psi$) is not a valid formula, otherwise ψ would be equivalent to the first order formula φ. Hence its negation $\varphi \wedge \neg\psi$ is satisfiable, i.e. there exists an S-structure $\mathfrak{B}$ such that $\mathfrak{B} \models \varphi$ and $\mathfrak{B} \models \neg\psi$.
The first part, $\mathfrak{B} \models \varphi$, means exactly that this $\mathfrak{B}|_{S_0}$ is m-isomorphic to $\mathfrak{A}|_{S_0}$ for an $\mathfrak{A}$ satisfying ψ. □

Internalizing Step 1

For Step 2, the passage from an m-isomorphism to a p-isomorphism, we have to express the statement of Step 1, i.e. (for given $S_0 \subseteq S$)

$$\exists\ S - str.\ \mathfrak{A}, \mathfrak{B} \quad s.t. \quad \mathfrak{A} \models \psi,\ \mathfrak{B} \models \neg\psi,\ \mathfrak{A}|_{S_0} \cong_m \mathfrak{B}|_{S_0}$$

internally in the language of our abstract logic.

More precisely we will construct a signature S^+ and an S^+-sentence $\gamma \in L(S^+)$ such that an S^+-structure satisfying γ consists of two S-structures, one satisfying ψ, the other $\neg\psi$, and an m-isomorphism between them.

Define $S^+ := S \cup \{U, V, W, P, <, I, G, f, c\}$ where U, V, W, P are unary relation symbols, $<, I$ are binary relations symbols, G is a ternary relation symbol, f a unary function symbol and c a constant symbol.

The statement of Step 1 gives us, for all $m \in \mathbb{N}$, an S^+-structure $\mathfrak{K}_m$:
The underlying set is $K_m := A \coprod B \coprod \{1, \dots, m\} \coprod P$, where A, B are the underlying sets of $\mathfrak{A}$, $\mathfrak{B}$, $P := \bigcup_{n=1}^m I_n$ (with I_n the set of n times extendable partial isomorphims $\mathfrak{A} \to \mathfrak{B}$).

– We interpret the symbols U as the subset A, V as the subset B, W as the subset $\{1, \dots, m\}$ and P as the subset which we already called P above.

– We interpret the binary symbol $<$ as the order relation on $W = \{1, \dots, m\}$ and $I \subseteq W \times P$ as the relation $I(n, p) :\Leftrightarrow p \in I_n$

– We interpret the ternary symbol G as $G \subseteq P \times A \times B$ where $G(p, a, b) :\Leftrightarrow a \in dom(p), p(a) = b$

– We interpret the function symbol f as the predecessor function on W and the constant symbol c as the maximal element m of W.

The S^+-structure $\mathfrak{K}_m$ satisfies the following first order sentences:

- $(W, <, f, c)$ is a total order with maximal element c and predecessor function f (where we set $f(0) = 0$)
- If $p \in P$, then p is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.
- If $n > 0$, $p \in I_n$ then, for any choice of $a \in U$ or $b \in V$ there is a $q \in I_{f(n)}$ extending p and with $a \in dom(q)$,, resp. $b \in Im(q)$.
- ψ^U, $(\neg\psi)^V$ hold — here we use the relativization property $\mathrm{Rel}(\mathcal{L})$ to build the formulas ψ^U, resp $(\neg\psi)^V$ saying that ψ, resp $\neg\psi$ hold on the sub-S-structures given by U, resp V. To form $\neg\psi$ we use that $\mathcal{L}$ contains Boolean connectives.

These are finitely many sentences (concrete fully formal sentences can be found in Ebbinghaus/Flum/Thomas) so we can form their conjunction, using $\mathrm{Bool}(\mathcal{L})$, and call the result γ.

Step 2

Prop.: For any finite $S_0 \subseteq S$ there are S-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \models_{\mathcal{L}} \psi$, $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$ and $\mathfrak{A}|_{S_0} \cong_p \mathfrak{B}|_{S_0}$.

Proof: Consider $\Gamma := \{\gamma\} \cup \{$ "W has at least m elements" $\mid m \in \mathbb{N} \}$. Since we have the S^+-structures $\mathfrak{K}_m$, all finite subsets are satisfiable. By $\mathrm{Comp}(\mathcal{L})$ the whole set is satisfiable, i.e. there exists an S^+-structure $\mathfrak{M}$ with $\mathfrak{M} \models \Gamma$.

This $\mathfrak{M}$ has $W \subseteq M$ an infinite totaly ordered set with maximal element, and has sub-S-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \models \psi$, $\mathfrak{B} \models \neg\psi$.

Define $I := \{p \in P \mid p \in I_{f^{(n)}(c)}$ for some $n \in \mathbb{N}\}$ (where $f^{(n)}$ means the n-fold application of the predecessor function). Every $p \in I$ is infinitely extendable (since W is infinite), so the the set I is a p-isomorphism $I \colon \mathfrak{A} \cong_p \mathfrak{B}$. $\square$

Step 3

We want to improve the result of Step 2 to saying the following:

For any finite $S_0 \subseteq S$ there are *countable* p-isomorphic S-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \vDash_{\mathcal{L}} \psi$, $\mathfrak{B} \vDash_{\mathcal{L}} \neg\psi$, and $\mathfrak{A}|_{S_0} \cong_p \mathfrak{B}|_{S_0}$. Being p-isomorphic and countable they will then be isomorphic.

For this it is of no use to apply $\mathrm{L\ddot{o}Sko}(\mathcal{L})$ to the structures $\mathfrak{A}$, $\mathfrak{B}$ directly: $\mathrm{L\ddot{o}Sko}(\mathcal{L})$ merely allows us to replace $\mathfrak{A}$, $\mathfrak{B}$ with countable, elementarily equivalent structures, but nothing guarantees that the two outcomes are p-isomorphic again.

Instead we take the structure $\mathfrak{M}$ from the proof of Step 2 and apply $\mathrm{LöSko}(\mathcal{L})$ to $\mathfrak{M}$ to get a countable model of our sentence γ from before. The two substructures $\mathfrak{A}$, $\mathfrak{B}$ will then also be countable.

The last thing that we have to take care of is that the ordered set that is the interpretation of W is still infinite (then we can, as in the proof of Step 2, get a p-isomorphism). To this end we enhance the signature S^+ by one more unary predicate Q. In the S^+-structure $\mathfrak{M}$ of Step 2 we interpret this as the set of predecessors of the maximal element c of W. Then $\mathfrak{M}$ is a model of the sentence $\theta := Q(c) \wedge \forall x(Q(x) \rightarrow ((f(x) < x) \wedge Q(f(x))))$ (which says that the set of predecessors of c is infinite).

Now, using $\mathrm{LöSko}(\mathcal{L})$, we pass to a countable model of $\gamma \wedge \theta$. The same moves as in Step 2 which defined a p-isomorphism, together with the countability, prove then the following proposition:

Prop. (Step 3): For any finite $S_0 \subseteq S$ there are S-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \vDash_{\mathcal{L}} \psi$, $\mathfrak{B} \vDash_{\mathcal{L}} \neg\psi$ and $\mathfrak{A}|_{S_0} \cong \mathfrak{B}|_{S_0}$.

Last step

This is not yet a contradiction, since we had to pass to a finite subsignature $S_0 \subseteq S$. We now show that this is enough, since the validity of ψ itself in an S-structure $\mathfrak{A}$ depends only on $\mathfrak{A}|_{S_0}$ for a finite subsignature S_0.

Lemma 1: Let $\Phi \subseteq L(S)$, $\varphi \in L(S)$, $\Phi \vDash_{\mathcal{L}} \varphi$. Then there exists a finite $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \vDash_{\mathcal{L}} \varphi$.

Proof: Choose $\neg\varphi$ using $\mathrm{Bool}(\mathcal{L})$. Then $\Phi \cup \{\neg\varphi\}$ is not satisfiable. Hence there is some finite $\Phi_0 \subseteq \Phi$ s.t. $\Phi_0 \cup \{\neg\varphi\}$ is not satisfiable. Hence $\Phi_0 \vDash_{\mathcal{L}} \varphi$. □

Lemma 2: Let $\psi \in L(S)$. Then there is a finite subset $S_0 \subseteq S$ such that for all S-structures $\mathfrak{A}$, $\mathfrak{B}$:
If $\mathfrak{A}|_{S_0} \cong \mathfrak{B}|_{S_0}$, then ($\mathfrak{A} \vDash_{\mathcal{L}} \psi$ iff $\mathfrak{B} \vDash_{\mathcal{L}} \psi$).

Proof: We consider a new signature $(S \cup \{U, V, f\})$ intended to talk about homomorphisms of S-structures: Given a homomorphism of S-structures $\mathfrak{A} \to \mathfrak{B}$ we can make an $(S \cup \{U, V, f\})$-structure $\mathfrak{M}$ with underlying set $A \coprod B$ s.t. $\mathfrak{M}|_A = \mathfrak{A}$, $\mathfrak{M}|_B = \mathfrak{B}$, U is a unary relation symbol interpreted as the subset A, V is a unary relation symbol interpreted as the subset B, and f is a binary relation symbol encoding the homomorphism between the two.

There is a set of sentences Φ of first order logic saying that f is an isomorphism between the S-structures $\mathfrak{M}^U$ and $\mathfrak{M}^V$. Clearly we have that

$$\Phi \vDash \psi^U \leftrightarrow \psi^V$$

(the right hand formula is built by using the Relativization and Boolean properties of $\mathcal{L}$, the entailment comes from the isomorphism property of $\mathcal{L}$).

(Proof continued)

By Lemma 1 there is a finite subset $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models \psi^U \leftrightarrow \psi^V$. As Φ, and hence Φ_0, consist of first order sentences, there is a finite subsignature $S_0 \subseteq S$ such that $\Phi_0 \subseteq L(S_0)$.

This subsignature S_0 has the desired property: If $\mathfrak{A}|_{S_0} \cong \mathfrak{B}|_{S_0}$, then we have an $(S \cup \{U, V, f\})$-structure which is a model of Φ_0. Because of $\Phi_0 \models \psi^U \leftrightarrow \psi^V$ we have $\mathfrak{A} \models \psi$ iff $\mathfrak{B} \models \psi$. $\square$

Proof of Lindström's theorem: Let $\mathcal{L}_{\omega\omega} < \mathcal{L}$, and assume that $\mathcal{L}$ is regular and satisfies $\mathrm{Comp}(\mathcal{L})$ and $\mathrm{LöSko}(\mathcal{L})$. From Steps 1 – 3 we get a signature S, a $\psi \in L(S)$ and for all finite subsignatures $S_0 \subseteq S$ we get S-structures $\mathfrak{A}$, $\mathfrak{B}$ such that

$$\mathfrak{A} \models_{\mathcal{L}} \psi, \quad \mathfrak{B} \models_{\mathcal{L}} \neg\psi, \quad \mathfrak{A}|_{S_0} \cong \mathfrak{B}|_{S_0} \qquad (*)$$

In particular this holds for the finite subsignature $S_0 \subseteq S$ of Lemma 2. But by Lemma 2 for this signature $(*)$ is a contradiction. $\square$

III. Other variants

A. More about $L_{\omega\omega}$

Sharpness of the result:

	Comp	LöSko
$\mathcal{L}_{\omega\omega}$	✓	✓
$\mathcal{L}^{2nd}$	✗	✗
$\mathcal{L}_{\kappa\lambda}$ in general	✗	✗
$\mathcal{L}_{\omega_1\omega}$	✗	✓
$\mathcal{L}_{\omega\omega}(Q_1)$	✓	✗
$\mathcal{L}_{\omega\omega}(Q^R)$	✗	✗(?)
$\mathcal{L}^{w2nd}$	✗	✓

Thus we can not drop the condition Comp or LöSko.

Sharpness of the result (cont.):

Define a logic $\mathcal{L}$ by
$L(S) := \{$2nd order sentences of the form $\exists X_1, \dots, X_n \psi$ where ψ contains no 2nd order quantifier $\}$
Satisfaction relation $\models_{\mathcal{L}}$ is that of $\mathcal{L}^{2nd}$.

Then:

(i) $\mathcal{L}$ is an abstract logic
(ii) $\mathcal{L}_{\omega\omega} < \mathcal{L}$
(iii) $\mathrm{LöSko}(\mathcal{L}), \mathrm{Comp}(\mathcal{L}), \mathrm{Repl}(\mathcal{L}), \mathrm{Rel}(\mathcal{L})$ hold
(iv) $\mathrm{Bool}(\mathcal{L})$ does not hold

Definition: $\mathcal{L}$ satisfies *countable compactness* if, given a *countable* $\Phi \subseteq L(S)$ then, if all finite subsets are satisfiable, it follows that Φ is satisfiable.

Theorem: $L_{\omega\omega}$ is the most expressive regular abstract logic satisfying *countable compactness* and Löwenheim-Skolem.

Theorem: $L_{\omega\omega}$ is the most expressive regular abstract logic satisfying countable compactness and the following implication (Karp property): If $\mathfrak{A} \cong_p \mathfrak{B}$ then $\mathfrak{A} \equiv_{\mathfrak{L}} \mathfrak{B}$ (i.e. the S-structures $\mathfrak{A}, \mathfrak{B}$ satisfy the same $L(S)$-sentences).

For this and the following see: Lindström, On characterizing elementary logic, in: Logical Theory and Semantic Analysis, Synthese Library Volume 63, 1974, pp 129–146

See also Flum, Characterizing logics, Chapter III of Barwise/Feferman, Model-theoretic logics, Springer 1985

Definition: $\mathcal{L}$ satisfies the *Tarski Union property* if, given a chain $\mathfrak{M}_0 \leq_{\mathcal{L}} \mathfrak{M}_1 \leq_{\mathcal{L}} \mathfrak{M}_2 \leq_{\mathcal{L}} \dots$ of $\mathcal{L}$-elementary extensions, the inclusion $\mathfrak{M}_n \leq_{\mathcal{L}} \bigcup_i \mathfrak{M}_i$ is an $\mathcal{L}$-elementary extension for all n.

Theorem (Lindström): $L_{\omega\omega}$ is the most expressive regular abstract logic satisfying compactness and the Tarski Union property.

There is a *property* $(+)$, roughly saying that one can take a sentence $\varphi \in L(S)$, replace all n-ary relation symbols in there by $(n+1)$-ary relation symbols, and then make it into a sentence again by binding the newly gained variable with a $\forall x$.

Theorem (Lindström): Suppose $\mathcal{L}_{\omega\omega} \leq \mathcal{L}$, $\mathcal{L}$ satisfies property $(+)$ and for all S-structures one has that $\mathfrak{A} \equiv \mathfrak{B}$ (elementary equivalence in $\mathcal{L}_{\omega\omega}$) implies $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$ (elementary equivalence in $\mathcal{L}$). Then $\mathcal{L}_{\omega\omega} \sim \mathcal{L}$.

Definition: From a relational signature S create a new signature S^+ by replacing each n-ary $P \in S$ with an $(n+1)$-ary P^+. From an S^+-structure $\mathfrak{A}$ and an $a \in A$ we get an S-structure $\mathfrak{A}^{(a)}$ with the same underlying set A by setting $P^{\mathfrak{A}^{(a)}} := \{(a, a_1, \ldots, a_n) \mid \mathfrak{A} \models P^+(a, a_1, \ldots, a_n)\}$.
Then $\mathcal{L}$ satisfies the property $(+)$, if for every $\varphi \in L(S)$ there is a $\varphi^+ \in L(S^+)$ such that for every S^+-structure $\mathfrak{A}$ one has: $\mathfrak{A} \models \varphi^+$ iff $\mathfrak{A}^{(a)} \models \varphi$ for all $a \in A$.

Remark: In the usual logics this is the following: From φ one obtains $\varphi'(x)$ by replacing $P(x_1, \ldots, x_n)$ with $P^+(x, x_1, \ldots, x_n)$ everywhere. Then $\varphi^+ = \forall x \varphi'(x)$. Indeed, property $(+)$ follows from some extra functoriality on signatures which allows to replace relation symbols in a formula with symbols of higher arity together with the existence of quantifiers.

Theorem (Lindström): Suppose $\mathcal{L}_{\omega\omega} \leq \mathcal{L}$, $\mathcal{L}$ satisfies $(+)$ and for all S-structures one has that $\mathfrak{A} \equiv \mathfrak{B}$ (elementary equivalence in $\mathcal{L}_{\omega\omega}$) implies $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$ (elementary equivalence in $\mathcal{L}$). Then $\mathcal{L}_{\omega\omega} \sim \mathcal{L}$.

Definition: $\mathcal{L}$ satisfies the *upward Löwenheim-Skolem property* if every $\varphi \in L(S)$ that has an infinite model, has an uncountable model.

Theorem (Lindström): Among the regular abstract logics with the property $(+)$, $\mathcal{L}_{\omega\omega}$ is the most expressive satisfying the upward and the downward Löwenheim-Skolem properties.

Effective versions: Suppose now that $L(S)$ is made of strings of symbols from some finite alphabet.

Definition: (a) $\mathcal{L}$ satisfies *completeness* if the set of valid sentences is recursively enumerable (i.e. there is some complete proof procedure).
(b) $\mathcal{L}$ *has effective negation and conjunction* if the negations and disjunction in the previous sense can be computed effectively.
(c) $\mathcal{L} \leq_{eff} \mathcal{L}'$ means that there is an effective procedure associating to each $\varphi \in \mathcal{L}$ a $\varphi' \in L'(S)$ which has the same models (i.e. is logically equivalent).

Theorem (Lindström): Suppose $\mathcal{L}_{\omega\omega} \leq_{eff} \mathcal{L}$, $\mathcal{L}$ has the downward Löwenheim property, is complete and has effective negation and conjunction. Then $\mathcal{L}_{\omega\omega} \sim_{eff} \mathcal{L}$

One can define what it means to have an *effective tableau method* for determining the valid sentences.

Theorem (Lindström): Suppose $\mathcal{L}_{\omega\omega} \leq_{eff} \mathcal{L}$, $\mathcal{L}$ has an effective tableau method and has effective negation and conjunction. Then $\mathcal{L}_{\omega\omega} \sim_{eff} \mathcal{L}$.

Remark: $\mathcal{L}_{\omega\omega}$ is the most expressive regular abstract logic satisfying a version of the Omitting types theorem.

(See: Flum, Characterizing logics, Thm 2.2.2. Originally: Lindström, Omitting uncountable types and extensions of elementary logic. Theoria 44 (1978), no. 3, 152–156 but only considering extensions of $\mathcal{L}_{\omega\omega}$ by quantifiers)

Open question: Is $\mathcal{L}_{\omega\omega}$ the most expressive regular abstract logic satisfying compactness and Craig interpolation?

See Väänänen, Lindström's theorem,
www.math.helsinki.fi/logic/opetus/lt/lindstrom_theorem1.pdf
Väänänen, The Craig Interpolation Theorem in abstract model theory, Synthese, 10/2008; 164(3):401-420.
Makowsky/Shelah, The theorems of Beth and Craig in abstract model theory I, Trans. Amer. Math. Soc. 256, 1979

III. Other variants

B. Other logics

Definition: A logic $\mathcal{L}$ is *bounded*, if for any S containing a binary relation $<$ and any $\varphi \in L(S)$ having only models with $<$ a well-ordering, there is an ordinal α, such that the order type of $<$ in any model is always smaller than α.

Theorem: $\mathcal{L}_{\infty\omega}$ is the most expressive regular logic that is bounded and has the Karp property (i.e. if $\mathfrak{A} \cong_p \mathfrak{B}$ then $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$).

See Flum, Characterizing Logics, Thm. 3.1. The Karp property is a substitute for downward Löwenheim-Skolem, the boundedness is a substitute for compactness.

Definition: (a) A logic $\mathcal{L}$ has *occurrence number* α, if α is the smallest cardinal such that for all S one has $L(S) = \{L(T) \mid T \subseteq S, |T| < \alpha\}$.
Notation: $oc(\mathcal{L})$
(b) Consider all $\varphi \in L(S)$ having only models with $<$ a well-ordering, and for which there is an ordinal α, such that the order type of $<$ in every model is smaller than α. The supremum of all α occurring thus is called *the well-ordering number*, $wo(\mathcal{L})$.

Theorem: $\mathcal{L}_{\kappa\omega}$ is the most expressive regular logic that is bounded, has the Karp property and has $oc(\mathcal{L}) \leq \kappa$ and $wo(\mathcal{L}) \leq \kappa$.

See Flum, Characterizing Logics, Thm. 3.2

Topological reformulation of Lindström's Theorem

Definition: For a signature S denote by $S - Str$ the class of S-structures. The *elementary topology* is the topology on $S - Str$ with the elementary classes $\mathrm{Mod}(\varphi) = \{\mathfrak{M} \mid \mathfrak{M} \vDash \varphi\}$ as open basis.

Because we have negation, it is also a closed basis (i.e. a clopen basis). It follows that the topology is *regular*, i.e. a closed set and a point outside of it can be separated by disjoint open sets.

Facts:

– The open sets are closed under isomorphism of S-structures (=:the topology is *invariant*).

– The reduction map $S_1 - Str \to S_0 - Str$ coming from an inclusion of signatures $S_0 \subseteq S_1$ is continuous (also "renamings").

Some reformulations:

Compactness theorem $\Leftrightarrow$ $S - Str$ is compact $\Leftrightarrow$ every ultrafilter has a limit $\Leftrightarrow$ Łoš's theorem

Downward Löwenheim-Skolem $\Leftrightarrow$ the countable S-structures are dense

Topological Lindström theorem (Caicedo): For each S let Γ_S be a regular, invariant topology on $S - Str$ such that the countable structures are dense, reduct and renaming maps are continuous, the Γ_S are compact and at least as fine as the elementary topology. Then the Γ_S *are* the elementary topologies.

See Caicedo, Lindström's Theorem for Positive Logics, a Topological View; in: Logic Without Borders: Essays on Set Theory, Model Theory, Philosophical Logic and Philosophy of Mathematics. De Gruyter. 73–90 (2015)

Lindström's theorem for positive logics

Definition: $\mathcal{L}$ is a weak extension of $\mathcal{L}$ ($\mathcal{L} \leq_w \mathcal{L}'$) :iff each sentence of $L(S)$ is equivalent to a theory of $L'(S)$. Write $\mathcal{L} \sim_w \mathcal{L}'$ for $\mathcal{L} \leq_w \mathcal{L}'$ and $\mathcal{L} \geq_w \mathcal{L}'$.

Consider logics without negation. Denote by $\mathrm{LöSko}_2$ the following version of downward Löwenheim-Skolem: A sentence that is true in all countable models of a theory is true in all models of that theory. We define a new topology on S-Str by declaring the classes $\mathrm{Mod}(\varphi)$ to be a sub-basis of *closed* classes.

Theorem (Caicedo): Any regular (now meaning: induces a regular topology on each class S-Str), compact logic with $\mathcal{L}_{\omega\omega} \leq_w \mathcal{L}$, having disjunctions and satisfying $\mathrm{LöSko}_2$ satisfies $\mathcal{L} \sim_w \mathcal{L}_{\omega\omega}$

Lindström theorems for modal logics

New issues arise for modal logics:

1. They are fragments of first order logic – we cannot import 1st order formulas in the proofs, as before.
2. They are interpreted in other structures, coming with their own notions of (partial) isomorphisms, back and forth etc.

Definition: (a) A *Kripke model* is an S-structure $\mathfrak{M}$ for the signature $S := \{A, R_1, R_2, R_3, \ldots\}$ (M the set of worlds, $A^{\mathfrak{M}}$ the accessibility relation, $R_i^{\mathfrak{M}}$ encodes a valuation for the variable x_i at each world)
(b) A *pointed Kripke model* is a pair $(\mathfrak{M}, w \in M)$
(c) An *abstract modal logic* is a pair $\mathcal{L} = (Fm_{\mathcal{L}}, \vDash_{\mathcal{L}})$ where $Fm_{\mathcal{L}}$ is a set ("$\mathcal{L}$-formulas") and $\vDash_{\mathcal{L}}$ is a relation between pointed Kripke models and $\mathcal{L}$-formulas.

Standing assumption: $\mathcal{L}$-formulas are invariant under isomorphism, $\mathcal{L}$ has Boolean operations, we have renaming and relativization.

Definition: (a) A *bisimulation* between Kripke models $\mathfrak{M}$, $\mathfrak{N}$ is a binary relation Z between M and N, such that:

(i) If wZv then $R_i^{\mathfrak{M}}(w) \Leftrightarrow R_i^{\mathfrak{N}}(v)$

(ii) If wZv and $wA^{\mathfrak{M}}w'$ there is a v' s.t. $vA^{\mathfrak{N}}v'$ and $w'Zv'$

(iii) If wZv and $vA^{\mathfrak{N}}v'$ there is a w' s.t. $wA^{\mathfrak{N}}w'$ and $w'Zv'$

(b) Two pointed Kripke models $(\mathfrak{M}, w)$, $(\mathfrak{N}, v)$ are *bisimilar* if there exists a bisimulation Z with wZv.

(c) A formula φ is *bisimulation invariant* if, given bisimilar $(\mathfrak{M}, w)$, $(\mathfrak{N}, v)$ one has $(\mathfrak{M}, w) \vDash \varphi \Leftrightarrow (\mathfrak{N}, v) \vDash \varphi$

(d) A logic is *bisimulation invariant* if all its formulas are.

A Lindström theorem for modal logic

Theorem(van Benthem, 2007): An abstract modal logic extending basic modal logic and satisfying compactness and bisimulation invariance is equally expressive as the basic modal logic K.

See ten Cate/Väänänen/van Benthem: Lindström theorems for fragments of first order logic, Logical Methods in Computer Science 5(3): 3 (2009)

Further results: Lindström theorem by S. Enqvist for Kripke frames axiomatizable by "strict first order Horn clauses" (2013), Lindström theorems for coalgebra semantics (Kurz/Venema, Enqvist), other results by de Rijke, Otto/Piro, Vuković, ...