# Lindström's Theorem

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# The statement

**Theorem (Lindström)**

*There is no logic that is more expressive than classical first order logic and that satisfies both the Compactness and the Löwenheim-Skolem properties.*

From: Per Lindström, On extensions of elementary logic, Theoria 35, p.1-11, 1969

# Abstract Logics

**Definition:** An *abstract logic* $\mathcal{L}$ consists of a function $L\colon$ signatures $\to$ sets and a binary relation $\models_{\mathcal{L}}$ betwen $S$-structures and elements of $L(S)$ (written $\mathcal{M} \models_{\mathcal{L}} \varphi$), such that

(a) If $S_0 \subseteq S_1$ then $L(S_0) \subseteq L(S_1)$

(b) If $\mathfrak{M} \models_{\mathcal{L}} \varphi$ and $\mathfrak{M} \cong \mathfrak{N}$ then $\mathfrak{N} \models_{\mathcal{L}} \varphi$

(c) If $S_0 \subseteq S_1$, $\varphi \in L(S_0)$ and $\mathfrak{M}$ is an $S_1$-structure, then $\mathcal{M} \models_{\mathcal{L}} \varphi$ iff $\mathcal{M}|_{S_0} \models_{\mathcal{L}} \varphi$

For $\varphi \in L(S)$ we write $\mathrm{Mod}_{\mathcal{L}}(\varphi) := \{\mathfrak{M} \in S - \mathrm{structures} \mid \mathfrak{M} \models \varphi\}$

# Abstract Logics: Examples

(1) First order logic with $L(S)$ and $\models$ as defined before.

# Abstract Logics: Examples

(2) The second order logic $\mathcal{L}^{2nd}$:

For $L^{2nd}(S)$-formulas we adopt the generation rules of first order $S$-formulas. Additionally we have *relation variables* of all arities and declare:

(a) If $X$ is an $n$-ary relation variable and $t_1, \dots, t_n$ are terms, then $X(t_1, \dots, t_n)$ is an $S$-formula

(b) If $\varphi$ is an $S$-formula, and $X$ is a relation variable, then $\exists X \varphi$ is an $S$-formula.

(c) An $L^{2nd}(S)$-sentence is a $L^{2nd}(S)$-formula without free variables.

Satisfaction relation: For first order formation rules as usual. Additionally declare for an $n$-ary relation variable:

$\mathfrak{M} \models_{\mathcal{L}^{2nd}} \exists X \varphi \;:\Leftrightarrow$ there is an $R \subseteq M^n$ such that $\mathfrak{M} \models_{\mathcal{L}^{2nd}} \varphi(R/X)$

# Abstract Logics: Examples

(3) The logics $\mathcal{L}_{\kappa\lambda}$:

For cardinals $\kappa \geq \lambda$ define the $L_{\kappa\lambda}(S)$-formulas as for first order logic, plus:
– for a set $\{\varphi_i \mid i \in I\}$, $|I| \leq \kappa$, one has a formula $\bigwedge \varphi_i$
– for a set of variables $\{x_i \mid i \in I\}$, $|I| \leq \lambda$ and a formula $\varphi$ one has a formula $\exists(x_i \mid i \in I)\varphi$.

Satisfaction relation: For first order formation rules as usual. Additionally
– $\mathfrak{M} \models_{\mathcal{L}_{\kappa\lambda}} \bigwedge \varphi_i \;:\Leftrightarrow \mathfrak{M} \models_{\mathcal{L}_{\kappa\lambda}} \varphi_i$ for all $i \in I$
– $\mathfrak{M} \models_{\mathcal{L}_{\kappa\lambda}} \exists(x_i \mid i \in I)\varphi \;:\Leftrightarrow$ there is $\{m_i \mid i \in I\} \subseteq M$ such that $\mathfrak{M} \models_{\mathcal{L}_{\kappa\lambda}} \varphi(m_i/x_i)$

1. Note that $\mathcal{L}_{\omega\omega}$ is classical first order logic.
2. One also allows the case $\kappa$ or $\lambda = \infty$ where one imposes no cardinality restriction.

# Abstract Logics: Examples

(4) $\mathcal{L}_{\omega\omega}(Q_1)$ := usual 1st order logic enhanced with the quantifier $Q_1$, interpreted as "there exist uncountably many"

(5) $\mathcal{L}_{\omega\omega}(Q^R)$ := usual 1st order logic enhanced with the *binary* quantifier $Q^R$, interpreted as
$\mathfrak{M} \vdash_{\mathcal{L}_{\omega\omega}(Q^R)} Q^R xy\,[\varphi(x), \psi(y)] :\Leftrightarrow card\{m \in M \mid \mathfrak{M} \vdash \mathcal{L}_{\omega\omega}(Q^R)\varphi(m)\} < card\{m \in M \mid \mathfrak{M} \vdash \mathcal{L}_{\omega\omega}(Q^R)\psi(m)\}$

(6) Weak second order logic $\mathcal{L}^{w2nd}$: Same syntax as $\mathcal{L}^{2nd}$ but relation variables are only interpreted as ranging over *finite* subsets of $M^n$.

# Abstract Logics: Non-example

*NOT* an example: start from a *2nd order signature* **S** containing relation/function/constant symbols as before, and additionally second order relation symbols interpreted as relations between *subsets* of the domain of interpretation.

There are obvious notions of **S**-structure, and of isomorphism of **S**-structures.

One can set up a language $L(\mathbf{S})$ from such a 2nd order signature **S** (best done using sorts) and define the obvious satisfaction relation between **S**-structures and $L(\mathbf{S})$-sentences (example: one can define the theory of topological spaces).

*Our logics are always based on first order signatures!*

# Expressivity of abstract logics

**Definition:** Let $\mathcal{L}$, $\mathcal{L}'$ be abstract logics.

(1) $\varphi \in L(S)$ and $\psi \in L'(S)$ are *logically equivalent* $:\Leftrightarrow$ $\mathrm{Mod}_{\mathcal{L}}(\varphi) = \mathrm{Mod}_{\mathcal{L}'}(\psi)$

(2) $\mathcal{L}' \geq \mathcal{L}$ ("$\mathcal{L}'$ *has at least the same expressive power* as $\mathcal{L}$") $:\Leftrightarrow$ for every $\varphi \in L(S)$ there is a $\psi \in L'(S)$ which is is logically equivalent to $\varphi$.

We write $\mathcal{L}' \sim \mathcal{L}$ (equal expressive power), if $\mathcal{L}' \geq \mathcal{L}$ and $\mathcal{L}' \leq \mathcal{L}$.
We write $\mathcal{L}' > \mathcal{L}$ if $\mathcal{L}' \geq \mathcal{L}$ and not $\mathcal{L}' \sim \mathcal{L}$.

# Expressivity of abstract logics

*Example 1:* Up to iso $\mathbb{R}$ is the only complete ordered field. In $\mathcal{L}^{2nd}$ we can hence characterize $\mathbb{R}$ up to isomorphism by adding to the theory of ordered fields the sentence
$\forall X((\exists x X(x) \wedge \exists y \forall z(X(z) \rightarrow z < y)) \rightarrow \exists y(\forall z(X(z) \rightarrow (z < y \vee z = y)) \wedge \forall x(x < y \rightarrow \exists z(x < z \wedge X(z)))))$
("every nonempty subset which is bounded above has a supremum")

By Löwenheim-Skolem we can not characterize $\mathbb{R}$ up to isomorphism in first order language. Hence $\mathcal{L}^{2nd} > \mathcal{L}_{\omega\omega}$.

*Example 2:* In $\mathcal{L}_{\omega_1\omega}$ we can characterize the class of fields of characteristic 0 by adding to the theory of fields the sentence
$\bigvee\{1+1=0, 1+1+1=0, 1+1+1+1+1=0, \ldots\}$

By Application 1 of the compactness theorem, there is no first order sentence characterizing fields of characteristic 0. Hence $\mathcal{L}_{\omega_1\omega} > \mathcal{L}_{\omega\omega}$.

# Properties of abstract logics

**Definition:** For an abstract logic $\mathcal{L}$ we abbreviate:

- $\mathrm{L\ddot{o}Sko}(\mathcal{L})$ ("$\mathcal{L}$ has the Löwenheim-Skolem property") $:\Leftrightarrow$ If $\varphi \in L(S)$ has a model, then it has a model which is at most countable.
- $\mathrm{Comp}(\mathcal{L})$ ("$\mathcal{L}$ has the compactness property") $:\Leftrightarrow$ If $\Phi \subseteq L(S)$ and every finite subset of $\Phi$ is satisfiable, then $\Phi$ is satisfiable.

# Properties of abstract logics

**Definition:** For an abstract logic $\mathcal{L}$ we abbreviate:

- $\mathrm{Bool}(\mathcal{L})$ ("$\mathcal{L}$ contains Boolean connectives") :$\Leftrightarrow$
  (1) For every $\varphi \in L(S)$ there is a $\chi \in L(S)$ such that for all $S$-structures $\mathfrak{M}$: $\mathfrak{M} \models \varphi \Leftrightarrow$ not $\mathfrak{M} \models \chi$
  (2) For every $\varphi, \psi \in L(S)$ there is a $\chi \in L(S)$ such that for all $S$-structures $\mathfrak{M}$: $\mathfrak{M} \models \chi \Leftrightarrow \mathfrak{M} \models \varphi$ and $\mathfrak{M} \models \psi$

*Example:* $\mathcal{L}_{\omega\omega}$ contains Boolean connectives: Take in (1) $\chi := \neg\varphi$ and in (2) $\chi := \varphi \wedge \psi$

# Properties of abstract logics

**Definition:** For an abstract logic $\mathcal{L}$ we abbreviate:

- $\mathrm{Repl}(\mathcal{L})$ ("$\mathcal{L}$ admits replacement of function symbols and constants by relation symbols"):
  From a signature $S$ we get a new signature $S^r$ by replacing $n$-ary function (resp. constant) symbols with $(n+1)$-ary (resp. unary) relation symbols.
  From an $S$-structure $\mathfrak{M}$ we get an $S^r$-structure $\mathfrak{M}^r$ by interpreting the new relation symbols as the graphs of the functions $f^{\mathfrak{M}}$.
  Then: $\mathrm{Repl}(\mathcal{L}) :\Leftrightarrow$ For every $\varphi \in L(S)$ there is a $\chi \in L(S^r)$ such that for all $S$-structures $\mathfrak{M}$ we have $\mathfrak{M} \vDash \varphi \Leftrightarrow \mathfrak{M}^r \vDash \chi$.

  *Example:* $\mathcal{L}_{\omega\omega}$ admits replacement; one can take $\chi$ as saying that the new relation symbols are graphs of functions that satisfy the corresponding statements of $\varphi$.

# Properties of abstract logics

**Definition:** For an abstract logic $\mathcal{L}$ we abbreviate:

- $\mathrm{Rel}(\mathcal{L})$ ("$\mathcal{L}$ admits relativization"):
  For an $S$-structure $\mathfrak{M}$ and an $S$-closed subset $A \subseteq M$ we get a sub-$S$-structure $\mathfrak{M}|_A$ with underlying set $A$.
  We also get an $S \cup \{U\}$-structure $\mathfrak{M}^{U \rightsquigarrow A}$ ($U$ a new unary relation symbol), with underlying set $M$, where $U$ is interpreted as the subset $A$.
  Then: $\mathrm{Rel}(\mathcal{L}) :\Leftrightarrow$ For every $\varphi \in L(S)$ there is a $\phi^U \in L(S \cup \{U\})$ such that $\mathfrak{M}|_A \models \varphi \Leftrightarrow \mathfrak{M}^{U \rightsquigarrow A} \models \phi^U$

  *Example:* $\mathcal{L}_{\omega\omega}$ admits relativization; one can take $\phi^U := \forall x \, (U(x) \rightarrow \phi)$

**Definition:** An abstract logic satisfying $\mathrm{Bool}$, $\mathrm{Repl}$ and $\mathrm{Rel}$ is called regular.

# Lindström's Theorem

**Theorem (Lindström's Theorem)**

*For a regular abstract logic* $\mathcal{L}$ *with* $\mathcal{L}_{\omega\omega} \leq \mathcal{L}$ *one has: If* $LöSko(\mathcal{L})$ *and* $Comp(\mathcal{L})$ *then* $\mathcal{L} \sim \mathcal{L}_{\omega\omega}$.

Equivalently: $\mathcal{L}_{\omega\omega}$ is the most expressive regular abstract logic having the Löwenheim-Skolem and Compactness properties.

# II. The proof

following Ebbinghaus/Flum/Thomas, Introduction to mathematical logic, Chapters XII/XIII

# Back and forth method

**Definition:** (a) A *partial isomorphism* between $S$-structures $\mathfrak{A}$, $\mathfrak{B}$ is an isomorphism between subsets of $A$ and $B$ respecting the relations/functions/constants. For a partial iso $p$, $dom(p)$ denotes the *domain* of $p$, and $rg(p)$ the *range* of $p$.

(b) An *m-isomorphism* is a sequence $I_1, \ldots, I_m$ of partial isomorphisms such that

(i) (forth-property) For every $p \in I_{n+1}$ and $a \in A$ there is a $q \in I_n$ with $q \supseteq p$ and $a \in dom(q)$

(ii) (back-property) For every $p \in I_{n+1}$ and $b \in B$ there is a $q \in I_n$ with $q \supseteq p$ and $b \in rg(q)$

If there is an $m$-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, we write $\mathfrak{A} \cong_m \mathfrak{B}$

**Proposition:** If $\mathfrak{A} \cong_m \mathfrak{B}$ then $\mathfrak{A}$ and $\mathfrak{B}$ satisfy exactly the same sentences of *quantifier rank* $\leq m$.

# Back and forth method

*Remark:* The case $m = \omega$ is also considered. If $\mathfrak{A} \cong_\omega \mathfrak{B}$, one says that $\mathfrak{A}$ and $\mathfrak{B}$ are *finitely isomorphic*.
*Theorem (Fraïssé):* $\mathfrak{A} \cong_\omega \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ are *elementary equivalent* (i.e. satisfy exactly the same first order sentences).

(c) A *p-isomorphism* is a set $I$ of partial isomorphisms such that
  (i) (forth-property) For every $p \in I$ and $a \in A$ there is a $q \in I$ with $q \supseteq p$ and $a \in dom(q)$
  (ii) (back-property) For every $p \in I$ and $b \in B$ there is a $q \in I$ with $q \supseteq p$ and $b \in rg(q)$

I.e. a $p$-isomorphism is an $\omega$-isomorphism in which all the sets $I_n$ are equal.
Notation: $\mathfrak{A} \cong_p \mathfrak{B}$

**Proposition:** If $\mathfrak{A} \cong_p \mathfrak{B}$ and $A$, $B$ are countable, then $\mathfrak{A} \cong \mathfrak{B}$

**1st outline of Lindström's proof:** Let $\mathcal{L}$ be a regular logic satisfying $\mathrm{LöSko}(\mathcal{L})$ and $\mathrm{Comp}(\mathcal{L})$. Assume that $\mathcal{L}_{\omega\omega} < \mathcal{L}$.

Then there exists a $\psi \in L(S)$ not equivalent to any first order sentence.

**1.** Show that for all $m \in \mathbb{N}$ there exist $S$-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \models_{\mathcal{L}} \psi$, $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$ and $\mathfrak{A} \cong_m \mathfrak{B}$.

**2.** Using $\mathrm{Comp}(\mathcal{L})$ we get $p$-isomorphic models $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \models_{\mathcal{L}} \psi$, $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$.

**3.** By $\mathrm{LöSko}(\mathcal{L})$ we can assume w.l.o.g. that $\mathfrak{A}$ and $\mathfrak{B}$ are *countable*. Then we have $\mathfrak{A} \cong \mathfrak{B}$ but $\mathfrak{A} \models_{\mathcal{L}} \psi$, $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$. This contradicts the isomorphism invariance of abstract logics!

# Expressing $m$-isomorphism type in 1st order logic

Let $S$ consist only of relation symbols. Let $L_r(S)$ denote the set of first order formulas containing at most the variables $x_0, \ldots, x_{r-1}$. Define $\Phi_r := \{\varphi \in L_r(S) \mid \varphi \text{ atomic}\} \cup \{\neg\varphi \in L_r(S) \mid \varphi \text{ atomic}\}$. Note that $\Phi_r$ is finite.

**Observation:** In first order logic one can define isomorphism types of finite relational structures.

*Proof:* For $\mathfrak{B}$ with $B = \{b_0, \ldots, b_{r-1}\}$ one can define $\varphi^0_{\mathcal{B}, b_0, \ldots, b_{r-1}}(x_0, \ldots, x_{r-1}) := \bigwedge\{\varphi \in \Phi_r \mid \mathfrak{B} \models \varphi(b_0, \ldots, b_{r-1})\}$. Now introduce constants for the elements of $B$, say that these are all elements and that $\varphi^0_{\mathcal{B}, b_0, \ldots, b_{r-1}}(b_0, \ldots, b_{r-1})$ holds. $\square$

Since $S$ is relational, from any $S$-structure $\mathfrak{B}$ and $b_0, \ldots b_{r-1} \in B$ we get a substructure $\{b_0, \ldots b_{r-1}\}$. Now use this to express the $m$-isomorphism type of relational structures.

# Expressing $m$-isomorphism type in 1st order logic

Remember:
$\varphi^0_{\mathcal{B},b_0,\dots,b_{r-1}} := \bigwedge\{\varphi \in \Phi_r \mid \mathfrak{B} \models \varphi(b_0,\dots,b_{r-1})\}$.

We define $L_{\omega\omega}(S)$-formulas $\varphi^n_{\mathcal{B},b_0,\dots,b_{r-1}}(x_0,\dots,x_{r-1})$ with the following property:

For any $S$-structure $\mathfrak{A}$ and $a_0,\dots,a_{r-1} \in A$ we have that if $\mathfrak{A} \models \varphi^n_{\mathcal{B},b_0,\dots,b_{r-1}}(a_1,\dots,a_{r-1})$ then $a_i \mapsto b_i$ defines a partial isomorphism from $\{a_0,\dots a_{r-1}\}$ to $\{b_0,\dots b_{r-1}\}$ which is $n$ times extendable.

Do this by induction on $n$:

$$\varphi^{n+1}_{\mathcal{B},b_0,\dots,b_{r-1}} := \forall x_r \bigvee\{\varphi^n_{\mathcal{B},b_0,\dots,b_{r-1},b} \mid b \in B\} \wedge \bigwedge\{\exists x_r \varphi^n_{\mathcal{B},b_0,\dots,b_{r-1},b} \mid b \in B\}$$

*Note:* For every $n$ there exist only finitely many $\varphi^n_{\mathcal{B},b_0,\dots,b_{r-1}}$ (induction). Hence $\varphi^{n+1}_{\mathcal{B},b_0,\dots,b_{r-1}}$ is a first order formula.

# Expressing $m$-isomorphism type in 1st order logic

The first order formula

$$\varphi^n_{\mathcal{B},b_0,\dots,b_{r-1}} := \forall x_r \bigvee\{\varphi^{n-1}_{\mathcal{B},b_0,\dots,b_{r-1},b} \mid b \in B\} \wedge \bigwedge\{\exists x_r \varphi^{n-1}_{\mathcal{B},b_0,\dots,b_{r-1},b} \mid b \in B\}$$

we have just defined is a formula in $r$ variables.

Given an $S$-structure $\mathfrak{A}$ and $a_0, \dots, a_{r-1} \in A$ then $\mathfrak{A} \vDash \varphi^n_{\mathcal{B},b_0,\dots,b_{r-1}}(a_0, \dots, a_{r-1})$ says that there exists a partial isomorphism $\mathfrak{A} \to \mathfrak{B}$ that sends $a_i$ to $b_i$ and is $n$ times extendable choosing arbitrary elements in the domain $A$ or in the image $B$.

In particular $\varphi^n_{\mathcal{B},\emptyset}$ is a sentence and $\mathfrak{A} \vDash \varphi^n_{\mathcal{B},\emptyset}$ implies that there exists an $n$-isomorphism $\mathfrak{A} \to \mathfrak{B}$.

Informally $\varphi^n_{\mathcal{B},\emptyset}$ says (about an $S$-structure where it is interpreted):

"*This structure is n-isomorphic to* $\mathfrak{B}$".

# Step 1. of the proof of Lindström's theorem

Let $\mathcal{L}$ be a regular logic satisfying $\mathrm{LöSko}(\mathcal{L})$ and $\mathrm{Comp}(\mathcal{L})$. Assume that $\mathcal{L}_{\omega\omega} < \mathcal{L}$.

Then there exists a $\psi \in L(S)$ not equivalent to any first order sentence.

By $\mathrm{Repl}(\mathcal{L})$ (allowing to replace function symbols with relation symbols) we can assume w.l.o.g. that $S$ contains only relation symbols. Remember that we want to prove the following:

**Proposition**(Step 1)**:** For all $m \in \mathbb{N}$ and all finite $S_0 \subseteq S$ there exist $S$-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \models_{\mathcal{L}} \psi$, $\mathfrak{B} \models_{\mathcal{L}} \neg\psi$ and $\mathfrak{A}|_{S_0} \cong_m \mathfrak{B}|_{S_0}$.

(Note that we had to pass to a finite subsignature for our $m$-isomorphism. This will not be a problem in the course of the proof of Lindström's theorem)

**Proposition**(Step 1)**:** For all $m \in \mathbb{N}$ and all finite $S_0 \subseteq S$ there exist $S$-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \vDash_{\mathcal{L}} \psi$, $\mathfrak{B} \vDash_{\mathcal{L}} \neg\psi$ and $\mathfrak{A}|_{S_0} \cong_m \mathfrak{B}_{S_0}$.

**Proof:** Let $S_0 \subseteq S$ be finite and $m \in \mathbb{N}$. Define $\varphi := \bigvee\{\varphi^m_{\mathfrak{A}|_{S_0},\emptyset} \mid \mathfrak{A} \vDash \psi\}$ ("this structure is $m$-isomorphic to $\mathfrak{A}|_{S_0}$ for an $\mathfrak{A}$ with $\mathfrak{A} \vDash \psi$")

There is an $\mathfrak{A}$ with $\mathfrak{A} \vDash \psi$, because otherwise $\psi$ would be logically equivalent to a contradiction (this is what it means to have no models), and hence a first order formula. So the above disjunction is non-empty. From the definition of the $\varphi^m_{\mathfrak{A}|_{S_0},\emptyset}$ one can also see that the disjunction is finite, so $\varphi$ is a first order sentence.

Clearly $\psi \to \varphi$ is a valid formula: If an $S$-structure $\mathfrak{A}$ satisfies $\mathfrak{A} \vDash \psi$, then it occurs in the disjunction and its $S_0$-reduction is $m$-isomorphic to itself.

On the other hand $\varphi \to \psi$ (i.e. $\neg\varphi \vee \psi$) is not a valid formula, otherwise $\psi$ would be equivalent to the first order formula $\varphi$. Hence its negation $\varphi \wedge \neg\psi$ is satisfiable, i.e. there exists an $S$-structure $\mathfrak{B}$ such that $\mathfrak{B} \vDash \varphi$ and $\mathfrak{B} \vDash \neg\psi$.
The first part, $\mathfrak{B} \vDash \varphi$, means exactly that this $\mathfrak{B}|_{S_0}$ is $m$-isomorphic to $\mathfrak{A}|_{S_0}$ for an $\mathfrak{A}$ satisfying $\psi$. $\square$